

# TASI Lectures on Supergravity and String Vacua in Various Dimensions

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## Abstract

These lectures aim to provide a global picture of the spaces of consistent quantum supergravity theories and string vacua in higher dimensions. The lectures focus on theories in the even dimensions 10, 8, and 6. Supersymmetry, along with with anomaly cancellation and other quantum constraints, places strong limitations on the set of physical theories which can be consistently coupled to gravity in higher-dimensional space-times. As the dimensionality of space-time decreases, the range of possible supergravity theories and the set of known string vacuum constructions expand. These lectures develop the basic technology for describing a variety of string vacua, including heterotic, intersecting brane, and F-theory compactifications. In particular, a systematic presentation is given of the basic elements of F-theory. In each dimension, we summarize the current state of knowledge regarding the extent to which supergravity theories not realized in string theory can be shown to be inconsistent.

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# 1 Introduction

## 1.1 Motivation

Quantum field theory is an incredibly successful framework for describing the fundamental processes underlying most observable physics. Quantities such as the anomalous magnetic dipole moment of the electron can be computed from first principles in quantum electrodynamics (QED), giving results that agree with experiment for up to 10 significant digits. While in quantum chromodynamics (QCD) many physically interesting questions cannot be addressed using perturbative calculations, the framework of field theory itself is believed to be adequate for describing most observed phenomena involving strong nuclear interactions. The difficulty in the case of QCD arises from the strong coupling constant, which necessitates nonperturbative treatment by methods such as lattice simulation. The standard model of particle physics, which underlies essentially all observed phenomena outside gravity, is itself a quantum field theory. And it is likely that whatever phenomena are discovered at the LHC will also be describable in terms of a quantum field theory, whether in terms of a Higgs scalar field, supersymmetry, new strongly coupled physics, or other more exotic possibilities.

Quantum field theory itself, however, does not place stringent constraints on what kinds of theories can be realized in nature. Field theories can be constructed with virtually any gauge symmetry group, and with a wide range of possible matter content consisting of particles transforming in various representations of the gauge group. While the absence of quantum anomalies in gauge symmetries, as well as other macroscopic consistency conditions, place some simple constraints on what is possible within a quantum field theory, the space of apparently consistent field theories is vast.

The one physically observed phenomenon that cannot be directly described in terms of quantum field theory is gravity. As discussed in other lectures in this school, in some cases quantum gravity has a dual description in terms of a quantum field theory in a lower dimension. Attempting to describe a diffeomorphism invariant theory in space-time of dimension 4 or greater in terms of a field theory in the same space-time, however, runs into well-known difficulties. Quantum field theory is defined in a fixed background space-time, and allowing the space-time metric and topology itself to fluctuate takes us outside the regime of applicability of standard field theory methods.

Constructing a consistent quantum theory of gravity has proven to be substantially more difficult than identifying a quantum theory describing the other forces in nature. String theory unifies gravity with quantum physics. From the point of view of the low-energy field theory, however, string theory requires an infinite sequence of massive fields to smooth the divergences encountered when the theory is coupled to gravity. While there are many different ways in which the extra dimensions of string theory can be “compactified” to give different low-energy field theories coupled to gravity in four dimensions, the space of such models still seems much smaller than the full space of 4D quantum field theories.

This leads to a fundamental question, which can be taken as motivation for these lectures:

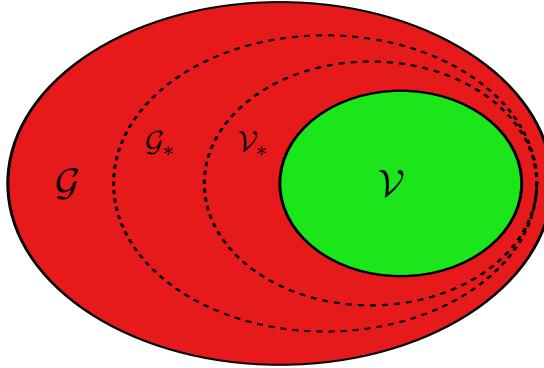


Figure 1: Venn diagram of supergravity theories. The set  $\mathcal{G}$  of apparently consistent quantum gravity theories (largest set, red + green regions) contains the set  $\mathcal{V}$  of known string vacua in any particular dimension (smallest set, green region). Intermediate sets  $\mathcal{G}_*$ ,  $\mathcal{V}_*$  denote the mathematically complete sets of consistent gravity theories and string vacua respectively. Set inclusions satisfy  $\mathcal{G} \supseteq \mathcal{G}_* \supseteq \mathcal{V}_* \supseteq \mathcal{V}$ .

**Motivating question:** *Does the inclusion of gravity place substantial constraints on the set of consistent low-energy quantum field theories?*

By low-energy here, we mean below the Planck scale.

To address this question we will attempt to give a global characterization of two general classes of theories, one of which is contained within the other.

$$\begin{aligned} \mathcal{G} &= \{\text{apparently consistent low-energy field theories coupled to gravity}\} \\ \mathcal{V} &= \{\text{low-energy theories arising from known string constructions}\} \end{aligned} \quad (1)$$

By “apparently consistent”, we mean that there is no known obstruction to constructing a consistent UV-complete quantum theory with the desired properties. By “known string constructions”, we mean a compactification of string theory in any of the regimes in which it is currently understood (*i.e.*, heterotic, type I/II, M-theory, F-theory, *etc.*). While some of these vacua cannot yet be described even perturbatively through a sigma model, and no background-independent fundamental formulation of string theory yet exists, we assume that there is a consistent quantum theory underlying string constructions. Pragmatically, this means that we assume that a few high-dimensional supergravity theories have consistent UV completions through string/M-theory, and consider the network of models that can be constructed by compactifying and adding branes and other features.

Assuming that string theory is indeed a consistent theory of quantum gravity, we have the inclusion

$$\mathcal{G} \supseteq \mathcal{V}. \quad (2)$$

The sets  $\mathcal{G}$  and  $\mathcal{V}$  as defined above are dependent upon our state of knowledge. We denote by  $\mathcal{G}_*$  the space of *actually* consistent gravity theories, and  $\mathcal{V}_*$  the complete space of possible string vacuum constructions (including those not yet discovered). The sets  $\mathcal{G}_*$  and  $\mathcal{V}_*$  are presumed to admit a mathematically precise definition, which has not yet been determined;

these sets do not depend upon our state of knowledge. We then have the series of inclusions

$$\mathcal{G} \supseteq \mathcal{G}_* \supseteq \mathcal{V}_* \supseteq \mathcal{V}. \quad (3)$$

By discovering new quantum consistency constraints, the space  $\mathcal{G}$  can be reduced, and by discovering new string vacuum constructions the space  $\mathcal{V}$  can be expanded<sup>1</sup>. As we discuss in these lectures, focusing on theories  $x \in \mathcal{G} \setminus \mathcal{V}$  can be a useful way of advancing knowledge. If  $x \in \mathcal{V}_*$  then there is a new string construction including  $x$ . If  $x \notin \mathcal{G}_*$  there is a physical constraint that makes  $x$  an inconsistent theory. And if  $x \in \mathcal{G}_* \setminus \mathcal{V}_*$  then there is a constraint particular to string theory that is violated by  $x$ . This discussion is closely related to Vafa's notion of the "swampland" [1, 2], which consists of gravity theories that appear to be consistent but which are not realized in string theory. In the notation defined above, the swampland is the set  $\mathcal{G} \setminus \mathcal{V}_*$ . We will sometimes refer to the set  $\mathcal{G} \setminus \mathcal{V}$  as the "apparent swampland". Part of the motivation for the approach taken here is the notion that we can learn a lot by trying to remove theories from the apparent swampland.

When the dimension of space-time is sufficiently large, and we restrict attention to gravity theories with supersymmetry, the constraints on  $\mathcal{G}$  are quite strong. In particular, as we discuss below in more detail, in 11 and 10 space-time dimensions the constraints are sufficiently strong that all supergravity theories not realized in string theory are known to be inconsistent. Thus, for the space of 11- and 10-dimensional supergravity theories we have  $\mathcal{G} = \mathcal{V}$ . It follows that

$$\mathcal{G}_* = \mathcal{V}_* \quad \text{for 10D and 11D supergravity.} \quad (4)$$

We say that *string universality* holds for supersymmetric theories of gravity in these dimensions, meaning that every theory not known to be inconsistent is realized in string theory [3].

In these lectures we focus attention on the discrete data characterizing the structure of supergravity theories: namely the field content and symmetries of the theory. The statement of string universality in (4) has been demonstrated in 10 and 11 dimensions at the level of these discrete structures. Any complete supergravity theory will contain a range of additional detailed structure encoded in continuous parameters such as coefficients of higher-derivative terms, the metric on moduli space, etc. Determining the uniqueness of such additional structure would refine our understanding of the relationship between the spaces  $\mathcal{G}$  and  $\mathcal{V}$ . While this represents a very interesting class of questions and challenges, we do not discuss these issues here.

In dimensions 10 and 11, as we shall describe in more detail below, there are only a handful of possible theories. As the dimension of space-time and the amount of supersymmetry

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<sup>1</sup>It is of course possible that some string vacuum constructions currently viewed as plausible by some parts of the community may be inconsistent, even if string theory is a fine theory. In this case we define  $\mathcal{V}$  to be the subset of hypothesized string vacua that are actually in  $\mathcal{V}_*$ ; in addition to the goals outlined in the text, the program described here can also be of assistance in determining which hypothetical string vacuum constructions are inconsistent (for example if a constraint on  $\mathcal{G}$  can be identified that is violated by some proposed vacua).

decreases, the constraints on possible theories become weaker, and the range of possible string constructions increases. We focus here on the classes of gravity theories with minimal supersymmetry in even dimensions. As the dimension decreases, the range of interesting physical phenomena also expands. In 8 dimensions, there are supergravity theories coupled to gauge theories with many different gauge groups. In 6 dimensions there can also be matter fields living in many different representations of the gauge group. In 4 dimensions, couplings such as Yukawa-type interactions arise, which increase the complexity of the theories significantly. These lectures focus primarily on dimensions 10, 8, and 6. By following possible string constructions through the decreasing range of dimensions, we can systematically introduce the new features and mechanisms giving rise to the phenomena that arise in each dimension. Thus, this approach gives a convenient pedagogical framework for introducing many fundamental aspects of different approaches to the construction of string vacua.

We do not discuss odd-dimensional supergravity theories here. While there are also interesting questions in odd dimensions, the global picture of theories in dimensions such as 9D is similar to the story described here for 8D theories. In even dimensions, there is also a somewhat richer structure both from the point of view of supergravity, where anomaly constraints can be stronger, and string theory, where F-theory provides a powerful nonperturbative framework for describing the space of string vacua. A systematic analysis of the discrete classes of string vacua with minimal supersymmetry in dimensions 9, 8, and 7 can be found in the work of de Boer *et al.* [4].

Another aspect of the theme of global structure of the space of theories is the connectivity of the space of theories. While different string theories, and different string vacua, may seem physically distinct, most of these theories are connected in various ways. Perturbative and nonperturbative duality symmetries relate different string constructions. In eight dimensions and six dimensions, most or all of the wide range of possible supersymmetric string vacua are different branches of a single theory, living in a continuous moduli space that is described in different regimes by different string constructions. In each of these dimensions we describe how the connectivity of this set of spaces arises, and the sense in which the diversity of string models fit into a single overarching theory.

We conclude this introductory section with a brief outline of the material presented in these lectures. In each of the dimensions 10, 8, and 6, we characterize the supergravity theories and introduce string vacuum constructions, then we compare what is known about the spaces  $\mathcal{G}$  and  $\mathcal{V}$ . Ten dimensions provides a good starting point for a systematic discussion of both supergravity theories and string vacua. After a brief summary of 11-dimensional supergravity, we describe the basic supergravity theories in 10 dimensions from which the lower-dimensional theories descend. On the supergravity side, anomaly constraints and the Green-Schwarz anomaly cancellation mechanism provide powerful tools for understanding the space  $\mathcal{G}$ . On the string side, basic objects such as strings and branes arise naturally and can be most easily understood in 10 dimensions. As mentioned above, in 10 dimensions  $\mathcal{G} = \mathcal{V}$ . Going down to 8 dimensions, we introduce the heterotic and F-theory approaches to string compactification. We describe how different gauge groups can be realized in these two types of constructions, and how the constructions are related. In each case, 8 dimensions

provides a natural domain in which to introduce the essential features of these classes of string vacua. In 8 dimensions we believe that we can identify  $\mathcal{V}$  with  $\mathcal{V}_*$ , but many theories still lie in  $\mathcal{G} \setminus \mathcal{V}$ . In six dimensions, as in 10 dimensions, anomaly constraints provide a powerful tool for understanding the space  $\mathcal{G}$  of gravity theories without known inconsistencies. We discuss additional complexities in the heterotic and F-theory vacuum constructions in 6D, and introduce intersecting brane models as another class of string vacua. F-theory vacua provide the largest set of 6D vacua constructions. Mathematical structure appearing in the anomaly cancellation conditions for any six-dimensional supergravity theory corresponds very closely to topological data for F-theory constructions. This enables us to identify a “bottom-up” map from low-energy theories to candidate string vacua in six dimensions. This map gives us a characterization of the embedding  $\mathcal{V} \subset \mathcal{G}$  and allows us to identify general features of models in the “apparent swampland”  $\mathcal{G} \setminus \mathcal{V}$ . F-theory provides a framework in which known supersymmetric 6D string vacua fit together into a single theory, with a moduli space connected through continuous deformations and phase transitions that can be understood in terms of F-theory geometry. We conclude the lectures with some comments on four-dimensional theories. While in four dimensions the spaces  $\mathcal{G}$  and  $\mathcal{V}$  are much larger and less well understood than in higher dimensions, some lessons from higher-dimensional supergravity and string constructions may be helpful in characterizing global aspects of the space of possible theories.

## 1.2 Background

These lectures are intended for an audience with some knowledge and experience with the basic principles and tools of quantum field theory. Some familiarity with elementary aspects of string theory is also helpful, though little specific technical knowledge is assumed. In the early part of the lectures, a number of concepts related to supersymmetry, supergravity, and perturbative string theory are reviewed briefly. The reader interested in more background on these topics should consult the textbooks of Green, Schwarz, Witten [5, 6] and Polchinski [7, 8]. The lectures generally follow the notation and conventions of Polchinski. A comprehensive overview of early work on supergravity theories in various dimensions can be found in the two-volume compilation by Salam and Sezgin [9].

The material covered in these lectures has evolved somewhat since the lectures were given, as some new results have clarified parts of the story. These written lecture notes integrate developments up to the time of writing (March 2011).

## 1.3 Supersymmetry

In these lectures we will restrict attention to theories with supersymmetry. Supersymmetry is a symmetry that relates bosons to fermions. We are interested here in theories where supersymmetry can be described as an extension of the Poincaré symmetry group that characterizes field theories in Minkowski space. At the level of the algebra of generators of the symmetry group, supersymmetry extends the Poincaré algebra by a set of fermionic

generators  $Q_\alpha, \bar{Q}_\alpha$  satisfying the anticommutation relations

$$\{Q_\alpha, \bar{Q}_\beta\} = 2P_\mu \Gamma_{\alpha\beta}^\mu, \quad (5)$$

where  $\alpha, \beta$  are spinor indices. The way in which bosonic and fermionic fields transform under an infinitesimal supersymmetry transformation parameterized by a spinor  $\epsilon$  takes the schematic form

$$\delta\phi \sim \bar{\epsilon}\psi, \quad \delta\psi \sim \Gamma^\mu \epsilon \partial_\mu \phi. \quad (6)$$

Some theories have multiple supersymmetries, with generators  $Q_\alpha^A$ , parameterized by an index  $A = 1, \dots, \mathcal{N}$ , where  $\mathcal{N}$  is the number of supersymmetries in the theory. In this case (5) generalizes to

$$\{Q_\alpha^A, \bar{Q}_\beta^B\} = 2\delta^{AB} P_\mu \Gamma_{\alpha\beta}^\mu \quad (7)$$

The supersymmetry algebra (7) can be extended by central charges that identify topological charges in the theory; for example, (7) can be extended to

$$\{Q_\alpha^A, \bar{Q}_\beta^B\} = 2\delta^{AB} P_\mu \Gamma_{\alpha\beta}^\mu + Z^{AB} \delta_{\alpha\beta}, \quad (8)$$

where the central charge  $Z^{AB}$  commutes with all other generators.

A description of supersymmetry and spinors in various dimensions is given in Appendix B of Volume 2 of Polchinski's text on string theory [8]. We will not use too many detailed aspects of supersymmetry and spinors in these lectures, but will assume some basic facts for which more detailed explanations can be found in that reference.

For a quantum field theory in flat space-time, supersymmetry is generally a global symmetry of the theory. If a theory of gravity has supersymmetry, however, then the symmetry becomes local. Just as translation symmetry in flat space-time becomes a symmetry under local diffeomorphisms in a generally covariant theory of gravity, in a supersymmetric generally covariant theory the spinor parameter  $\epsilon$  itself becomes a general space-time dependent function. A theory of gravity with local supersymmetry is called a *supergravity* theory. Each supergravity theory contains  $\mathcal{N}$  massless spin 3/2 *gravitino* fields  $\psi_{\mu\alpha}^A$  that are partners of the graviton  $g_{\mu\nu}$ .

A primary reason for considering only supersymmetric theories of gravity in these lectures is that supersymmetry imposes additional structure that makes both gravity and string theory easier to analyze and to understand. There are, however, physical reasons to be interested in supersymmetric theories, both from the point of view of low-energy phenomenology, and from the top-down point of view of string theory.

From the phenomenological point of view, supersymmetry has a number of desirable features. Supersymmetry modifies the renormalization group equations so that the strong, weak and electromagnetic couplings appear to unify at a high scale. Supersymmetry protects the mass of the scalar Higgs, giving a possible solution to the “hierarchy” problem. Supersymmetry also provides a natural candidate for dark matter.

From the string theory point of view, supersymmetry plays a crucial role in removing tachyonic instabilities from the theory at the Planck scale. While it is possible that some

intrinsically non-supersymmetric versions of string theory can be made mathematically consistent, string theory is best understood as a supersymmetric theory of quantum gravity in 10 dimensions.

While understanding the space of supersymmetric theories is a rewarding enterprise in its own right, which may also give new insights into the structure of non-supersymmetric theories, the most optimistic reason to study supersymmetric theories is the possibility that supersymmetry is manifest in our world at an experimentally accessible energy scale. The simplest framework in which this occurs can be analyzed following the assumption that physics can be split into two different energy scales. Under this assumption, at and below some scale  $\Lambda$ , all of the relevant physics describing our world can be characterized by a supersymmetric quantum field theory coupled to (classical) gravity. Above the scale  $\Lambda$ , supergravity and/or string theory are needed to describe physics at energies up to the Planck scale. If this assumption is correct, it means that quantum gravity will play a phenomenological role primarily in determining which supersymmetric QFT's can arise at the intermediate scale  $\Lambda$ . If, on the other hand, this assumption is incorrect and supersymmetry is broken at the Planck scale, making any progress in understanding the connection between quantum gravitational consistency and low-energy physics will be extremely challenging. From a theoretical point of view, one of the most exciting consequences of the discovery of supersymmetry at the LHC would be the confirmation that this separation of scales exists, guaranteeing that supersymmetry can be used in efforts to understand physics up to the Planck scale.

## 1.4 Supergravity in 11 dimensions

We are interested, therefore, in understanding supersymmetric theories of gravity (supergravity), which may also include gauge fields and various kinds of matter. In higher dimensions, supergravity theories are quite constrained. For dimensions  $D > 11$ , any representation of the Clifford algebra associated with gamma matrices of the relativistic symmetry group has dimension 64 or greater. This leads to massless particles related to the graviton by supersymmetry that have spin greater than 2. No interacting theories of this kind are known in dimensions above 11, and it is believed (though perhaps not rigorously proven) that the highest dimension in which a supersymmetric theory of gravity can exist is 11 dimensions.

In 11 dimensions, there is a unique supersymmetric theory of gravity. This theory has one supersymmetry ( $\mathcal{N} = 1$ ), and 32 supercharges  $Q_\alpha$  carrying an index in the 32-dimensional spinor representation of  $SO(1, 10)$ . The massless fields in the theory describe particles that are in the *supermultiplet* of the graviton; *i.e.*, states that are related to the graviton by acting with the supersymmetry generators. These fields include:

$g_{\mu\nu}$ : the graviton (quantum of fluctuation in the space-time metric)

The graviton is symmetric and traceless, with  $(9 \times 10)/2 - 1 = 44$  degrees of freedom.

$C_{\mu\nu\lambda}$ : an antisymmetric 3-form field (analogous to a gauge field  $A_\mu$  but with more indices)

The 3-form field has  $(9 \times 8 \times 7)/6 = 84$  degrees of freedom.

$\psi_{\mu\alpha}$ : the gravitino, with 128 degrees of freedom.

In each case the number of degrees of freedom can be understood by considering the appropriate representation of the  $SO(9)$  little group for massless states in  $SO(1, 10)$ , in a fashion directly analogous to the standard analysis of states in 4D QFT.

The low-energy action for the bosonic fields of 11-dimensional supergravity is given by

$$S = \frac{1}{2\kappa_{11}^2} \left[ \int \sqrt{g} (R - \frac{1}{2}|F|^2) - \frac{1}{6} \int C \wedge F \wedge F \right], \quad (9)$$

where  $\kappa_{11}$  is the 11-dimensional Newton constant, and the field strength  $F$  is a 4-form given by

$$F^{(4)} = dC^{(3)}. \quad (10)$$

This is a theory of pure supersymmetric gravity, with no conventional gauge symmetries or matter content (though the 3-form field  $C$  does have a higher-index version of an abelian gauge symmetry). For further details and references on this theory, the reader is referred to Polchinski's text [8].

From the point of view of these lectures, the significant feature of supergravity in 11 dimensions is that there is only one possible theory, at least at the discrete level of field content and symmetries. Thus,

$$\mathcal{G}^{(11)} = \{M_{11}\} \quad (11)$$

where  $M_{11}$  is the supergravity theory with bosonic action (9). Although this theory does not itself contain strings, it can be viewed as the strong coupling limit of a ten-dimensional string theory [10]. Thus, this theory is included in the general space  $\mathcal{V}$  of “known string vacua”, and in 11 dimensions we have

$$\mathcal{G}^{(11)} = \mathcal{V}^{(11)} \quad \Rightarrow \quad \mathcal{G}_*^{(11)} = \mathcal{V}_*^{(11)}. \quad (12)$$

So string universality holds in 11 dimensions. The quantum theory of 11-dimensional supergravity is often referred to as “M-theory”. As we discuss below, M-theory can also be described in the light-cone gauge through quantization of the membrane in the 11-dimensional theory, or alternatively in terms of pointlike branes in a 10D theory.

## 2 Supergravity and String Vacua in Ten Dimensions

We now consider supergravity theories in ten dimensions, and their UV completions through string theory. In ten dimensions, there are theories with one or two supersymmetries. We begin with a brief summary of the gravity theories with two supersymmetries, before considering the theories with one supersymmetry in more detail. We then discuss the string realization of these theories and the relationship between  $\mathcal{G}$  and  $\mathcal{V}$ . Again, further details on most of the material in this section can be found in Polchinski [8].

In 10 dimensions we can define an 11th gamma matrix

$$\Gamma^{11} = \prod_{\mu=0}^9 \Gamma^\mu \quad (13)$$

analogous to  $\gamma^5$  in four dimensions. As in four dimensions,  $\Gamma^{11}$  has eigenvalues  $\pm 1$  corresponding to Weyl spinors of fixed chirality. In dimensions of the form  $D = 4k + 2$ , the Weyl representations are self-conjugate. In ten dimensions it is also possible to impose a Majorana (reality) condition. The Lorentz group in 10D thus has two distinct real 16-dimensional representations **16**, **16'** corresponding to Majorana-Weyl chiral spinors. The different supergravity theories in 10D have different choices of spinor representations for the supersymmetry generators  $Q_\alpha^A$ .

## 2.1 $\mathcal{N} = 2$ supergravity in ten dimensions

### 2.1.1 Type IIA supergravity

The type IIA  $\mathcal{N} = 2$  supergravity theory in ten dimensions has two supersymmetries of opposite chirality, generated by  $Q^1 \in \mathbf{16}, Q^2 \in \mathbf{16}'$ . The theory has bosonic fields

$$g_{\mu\nu}, B_{\mu\nu}, \phi \quad (14)$$

living in an  $\mathcal{N} = 1$  supersymmetry multiplet. The field  $B_{\mu\nu}$  is an antisymmetric two-form field, and the field  $\phi$  is a scalar (the *dilaton*). The IIA theory has additional bosonic fields described by a 1-form and 3-form

$$A_\mu, C_{\mu\nu\lambda}. \quad (15)$$

From the nature of the stringy origin of these fields, they are referred to as *Ramond-Ramond*, or *R-R* fields.

Counting degrees of freedom, the bosonic fields in the IIA theory have 128 components. In accord with supersymmetry, this is the same as the number of fermionic degrees of freedom, which are contained in a pair of Majorana-Weyl gravitinos (**56** + **56'**) and a pair of Majorana-Weyl spinors<sup>2</sup> (**8** + **8'**). Because there is one supersymmetry of each type, the theory is non-chiral, with one spinor and one gravitino of each chirality.

The IIA theory is directly related to 11-dimensional supergravity by *compactification* on a circle  $S^1$ . Wrapping the 11D theory described by the action (9) on a circle of radius  $R$ , as  $R$  becomes small the momentum (Kaluza-Klein) modes on the extra circle become very massive, and in the low-energy limit the zero modes of the theory combine into the fields of the IIA supergravity theory. The dilaton, for example, comes from the component of the metric tensor in 11D with both indices wrapped in the compact direction  $g_{11\ 11} \rightarrow \phi$ . The

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<sup>2</sup>Note that the Majorana-Weyl spinors characterizing the supersymmetry generators transform under the relativistic symmetry group  $SO(1, 9)$ , while the on-shell degrees of freedom in a massless spinor field transform under the little group  $SO(8)$ ; this explains the discrepancy between the 16 and 8 real degrees of freedom in these representations.

correspondence between the 11D and 10D degrees of freedom for all the bosonic fields is given by

$$\begin{array}{ccccccc}
\text{IIA :} & \text{11D SUGRA} & g_{\mu\nu}^{(11)} & g_{\mu 11}^{(11)} & g_{11 11}^{(11)} & C_{\mu\nu\lambda}^{(11)} & C_{\mu\nu 11}^{(11)} \\
& \downarrow (S^1) & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& \mathbb{R}^{1,9} & g_{\mu\nu} & A_\mu & \phi & C_{\mu\nu\lambda} & B_{\mu\nu}
\end{array} \quad (16)$$

### 2.1.2 Type IIB supergravity

Type IIB supergravity in ten dimensions is similar to the type IIA theory in some ways, but is distinguished by having two supersymmetries of the same chirality,  $Q^1, Q^2 \in \mathbf{16}$ . The type IIB theory again contains the  $\mathcal{N} = 1$  multiplet of fields  $g_{\mu\nu}, B_{\mu\nu}, \phi$ , as well as a Ramond-Ramond axion  $\chi$ , two-form  $\tilde{B}_{\mu\nu}$ , and self-dual 4-form field  $D_{\mu\nu\lambda\sigma}^+$  whose massless states transform in the chiral **70** representation of the little group  $SO(8)$ . In this theory there are two gravitinos with the same chirality ( $2 \times \mathbf{56}$ ), and two spinors of identical chirality ( $2 \times \mathbf{8}'$ ). The self-dual field  $D^+$  makes it difficult to write a local Lagrangian for this theory, but classical equations of motion for the fields can be written in an unambiguous fashion. The type IIB supergravity theory has a classical global symmetry under  $SL(2, \mathbb{R})$ , under which the two-form fields  $B, \tilde{B}$  rotate into one another as a doublet representation.

The type IIA and IIB supergravity theories are uniquely fixed by the supersymmetries of the theory, at least in terms of the field content and low-energy equations of motion.

## 2.2 $\mathcal{N} = 1$ supergravity in ten dimensions

We now turn to ten-dimensional supergravity theories with only one supersymmetry,  $Q_\alpha \in \mathbf{16}$ . There are two kinds of supersymmetry multiplet that can appear in such theories; in addition to the gravity multiplet, there is a vector multiplet containing a vector field and a spinor called the *gaugino* in the **8** representation of  $SO(8)$ . The field content of the multiplets, and the associated numbers of bosonic and fermionic degrees of freedom, can be summarized as

$$\text{10D } \mathcal{N} = 1 \text{ multiplets} \quad \left\{ \begin{array}{lll} \text{SUGRA} & g_{\mu\nu}, B_{\mu\nu}, \phi, \psi_{\mu\alpha}, \zeta_\alpha & [64 + 64 \text{ DOF}] \\ \text{vector} & A_\mu, \lambda_\alpha & [8 + 8 \text{ DOF}] \end{array} \right. \quad (17)$$

Note that the spinor  $\zeta_\alpha$  in the gravity multiplet is in the **8'** of  $SO(8)$  and has the opposite chirality to any gauginos.

Classically,  $\mathcal{N} = 1$  supergravity can be coupled to a set of vector fields  $A_\mu^a$  realizing any abelian or nonabelian gauge group. The action for such a theory is

$$S \sim \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{g} \left[ e^{-2\phi} \left( R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H|^2 \right) - \frac{e^{-\phi}}{g_{\text{YM}}^2} F_{\mu\nu}^a F^{a\mu\nu} \right] \quad (18)$$

where  $g_{\text{YM}}$  is the Yang-Mills coupling constant,  $F$  is the usual field strength of the abelian or nonabelian gauge group, and the three-form field strength  $H$  is related to the two-form

$B$  field through

$$H = dB - \omega_Y \quad d\omega_Y = \sum_a F^a \wedge F^a. \quad (19)$$

The additional contribution of the Chern-Simons term  $\omega_Y$  to  $H$  is a consequence of supersymmetry [11, 12]; this term produces a coupling of the form  $BF^2$  that is related through supersymmetry to the gauge field kinetic term. We now discuss the conditions for quantum consistency of supergravity theories in 10D.

### 2.3 Anomalies in ten-dimensional supergravity theories

Not every theory that admits a classical Lagrangian formulation is quantum mechanically consistent. Classical symmetries can be broken by quantum anomalies. The most well-known example is the chiral anomaly in four-dimensional gauge theories, in which the current associated with a chiral symmetry develops a quantum correction at one loop. The non-vanishing of the divergence of the chiral current  $\partial_\mu j^{5\mu} \sim F \wedge F$  amounts to a quantum breaking of the chiral symmetry. The anomaly appears because the quantum theory cannot be regulated or made UV complete without breaking the symmetry. Anomalies can be understood in terms of one-loop corrections where the failure of a regulator to preserve the symmetry is manifest. Alternatively, anomalies can be understood in terms of a failure of the measure in the path integral to respect the classical symmetry of the theory

$$\int d\psi d\bar{\psi} \neq \int d\psi' d\bar{\psi}'. \quad (20)$$

In general, anomalies have a topological origin and can be related to an appropriate mathematical index theorem. For a good introduction to anomalies and more details on aspects relevant to these lectures, see Harvey's 2003 TASI lectures [13].

If a global symmetry in a theory becomes anomalous, such as the chiral symmetry in some 4D gauge theories, then it simply means that the symmetry is broken quantum-mechanically. The theory can still be a consistent quantum theory. If, however, a local symmetry is anomalous, it is generally impossible to find a consistent quantum completion of the theory. For example, if a theory contains a  $U(1)$  gauge symmetry that is anomalous, then the theory cannot be fixed unless an additional degree of freedom (such as a Stückleberg field) combines with the  $D - 2$  degrees of freedom of the massless gauge field to complete the necessary set of  $D - 1$  degrees of freedom for a massive gauge field.

Similarly, if local diffeomorphism invariance is broken by a quantum anomaly, there is no way to make the theory consistent as a quantum theory of gravity. As mentioned above, Weyl representations are self-conjugate in dimensions of the form  $D = 4k + 2$ . This means that particles and anti-particles have the same chirality, so that chiral fermions can contribute to gravitational and mixed gauge-gravitational anomalies as well as to purely gauge anomalies. The detailed form of such anomalies in ten dimensions was worked out in a classic paper by Alvarez-Gaume and Witten [14]. In ten dimensions, anomalies arise from hexagon diagrams

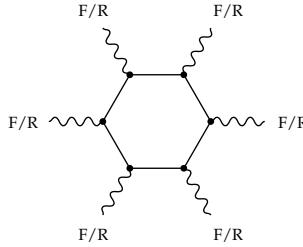


Figure 2: Hexagon diagrams give rise to gravitational, gauge, and mixed gauge-gravitational anomalies in ten dimensions.

with external gauge fields or gravitons (see Figure 2). The anomaly structure of any theory in  $D$  dimensions can be encoded in a  $(D+2)$ -form *anomaly polynomial*  $\hat{I}$ . Alvarez-Gaume and Witten showed that the respective contributions of  $n$  gaugino spinor fields (8), the gravitino field (56), and a self-dual 4-form field (70) to the 10D anomaly polynomial are

$$\begin{aligned} \hat{I}_{(8)} &= -\frac{\text{Tr}(F^6)}{1440} + \frac{\text{Tr}(F^4)\text{tr}(R^2)}{2304} - \frac{\text{Tr}(F^2)\text{tr}(R^4)}{23040} - \frac{\text{Tr}(F^2)[\text{tr}(R^2)]^2}{18432} \\ &\quad + \frac{n \text{tr}(R^6)}{725760} + \frac{n \text{tr}(R^4)\text{tr}(R^2)}{552960} + \frac{n [\text{tr}(R^2)]^3}{1327104} \end{aligned} \quad (21)$$

$$\hat{I}_{(56)} = -495 \frac{\text{tr}(R^6)}{725760} + 225 \frac{\text{tr}(R^4)\text{tr}(R^2)}{552960} - 63 \frac{[\text{tr}(R^2)]^3}{1327104} \quad (22)$$

$$\hat{I}_{(70)} = 992 \frac{\text{tr}(R^6)}{725760} - 448 \frac{\text{tr}(R^4)\text{tr}(R^2)}{552960} + 128 \frac{[\text{tr}(R^2)]^3}{1327104} \quad (23)$$

In the expression for  $\hat{I}_{(8)}$ , the terms containing  $F$  arise from the coupling of the gaugino spinor to the gauge field. The notation  $\text{Tr}$  indicates that these traces are evaluated in the adjoint representation of the gauge group. Note that for abelian vector fields, these terms must vanish, as the adjoint representation of  $U(1)$  is trivial.

The anomaly contributions from eqs. (21-23) must be summed over all chiral fields in a theory to determine the total anomaly polynomial. The condition that the total anomaly must cancel is a very strong condition on 10D gravity theories.

Let us first consider the anomaly contributions from the two  $\mathcal{N} = 2$  theories. The type IIA supergravity theory is non-chiral; each chiral field has a counterpart with the opposite chirality. So in this theory all anomalies cancel.

The type IIB theory, on the other hand, has a more complicated set of contributions. The two gravitinos contribute two factors of  $\hat{I}_{56}$ . The self-dual four-form field  $D^+$  contributes one factor of  $\hat{I}_{70}$ . The two spinors in the  $8'$  contribute as two gauginos in (21), but with the opposite sign and with no gauge contribution. The total anomaly for the IIB theory is then

$$-2\hat{I}_8(F \rightarrow 0, n \rightarrow 1) + 2\hat{I}_{56} + \hat{I}_{70} = 0. \quad (24)$$

Thus, the type IIB supergravity theory can be a quantum consistent theory through a rather intricate cancellation.

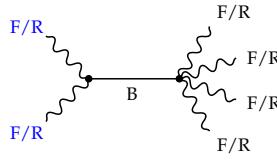


Figure 3: The Green-Schwarz mechanism: A tree diagram describing exchange of a  $B$  field can cancel the anomalous part of the one-loop hexagon diagram in ten dimensions in special cases.

## 2.4 The Green-Schwarz mechanism

From the above analysis, it seems that all  $\mathcal{N} = 1$  theories must be anomalous. Consider in particular the terms in  $\hat{I}_{(8)}, \hat{I}_{(56)}$  proportional to  $\text{tr}R^6$  and  $[\text{tr}(R^2)]^3$ . Since the spinor  $\zeta_\alpha$  in the gravity multiplet is in the  $\mathbf{8}'$  and contributes to the  $R^6$  term with the opposite sign of a gaugino, the  $\text{tr}R^6$  terms can only cancel if  $n = 496$ . So the gauge group must have dimension 496. But by a similar argument, the  $[\text{tr}(R^2)]^3$  terms cannot cancel unless  $n = 64$ . So both terms cannot cancel simultaneously.

This seems to doom all  $\mathcal{N} = 1$  theories of supergravity in ten dimensions. But, at the time of Alvarez-Gaume and Witten's analysis it was already known that a string theory, known as the type I string, exists and corresponds at low energies to an  $\mathcal{N} = 1$  supergravity theory in ten dimensions with gauge group  $\text{SO}(32)$ . Green and Schwarz analyzed this string theory carefully and showed that it evades the apparent anomaly constraint through a feature now known as the *Green-Schwarz mechanism* for anomaly cancellation [15].

The key to the Green-Schwarz mechanism is the realization that the anomalous gauge variation associated with 1-loop diagrams can be canceled by tree-level diagrams when higher-order terms that are themselves not gauge invariant are added to the action. To implement this mechanism in the case of a nonabelian gauge group, the field strength (19) must be enhanced at higher orders in the derivative expansion by a Chern-Simons term in the spin connection

$$H = dB - \omega_Y + k\omega_R, \quad d\omega_R = \text{tr}R^2, \quad (25)$$

with  $k$  a constant factor. The two-form  $B$  must then transform as  $\delta B = \text{Tr}(\Lambda F) - \text{tr}(\Theta R)$  where  $\delta A = d\Lambda$ , and the connection 1-form transforms through  $\delta\omega_1 = d\Theta$ . The anomaly can now be cancelled by a tree diagram (see Figure 3) in which a  $B$ -field is exchanged, when a “Green-Schwarz” term

$$\Delta S \sim \int B \wedge X_8(F, R) \quad (26)$$

is added to the action such that the anomaly can be expressed in the factorized form

$$\hat{I} = Y_4(F, R)X_8(F, R), \quad Y_4 = \text{tr}R^2 - \frac{1}{30}\text{Tr}F^2, \quad (27)$$

where the constant  $k$  has taken the value  $k = 30$ .

For a general nonabelian gauge group of dimension  $n$ , the total anomaly can be rearranged to take the form

$$\hat{I}_{12} = \frac{1}{1440} \left( -\text{Tr}F^6 + \frac{1}{48}\text{Tr}F^4\text{Tr}F^2 - \frac{(\text{Tr}F^2)^3}{14400} \right) \quad (28)$$

$$+ (n - 496) \left( \frac{\text{tr}R^6}{725760} + \frac{\text{tr}(R^4)\text{tr}(R^2)}{552960} + \frac{[\text{tr}(R^2)]^3}{1327104} \right) \quad (29)$$

$$- \frac{Y_4 X_8}{768},$$

where

$$X_8 = \text{tr}R^4 + \frac{(\text{tr}R^2)^2}{4} - \frac{(\text{Tr}F^2)(\text{tr}R^2)}{30} + \frac{\text{Tr}F^4}{3} - \frac{(\text{Tr}F^2)^2}{900}. \quad (30)$$

Thus, the Green-Schwarz mechanism can cancel the anomaly for a nonabelian group of dimension 496 precisely when the first term vanishes. There are exactly two nonabelian groups for which  $\text{Tr}F^6$  can be expressed as the necessary combination of  $\text{Tr}F^4\text{Tr}F^2$  and  $(\text{Tr}F^2)^3$ . These two groups are  $SO(32)$  and  $E_8 \times E_8$ .

Now consider a gauge group with abelian factors. Because supersymmetry requires (19) to contain a contribution from the Chern-Simons term of each abelian factor, there is a coupling in the action of the form  $BF^2$  for each  $U(1)$  factor in the theory. For the pure gravitational anomaly to vanish, the Green-Schwarz term (26) must be added to the action. But the  $R^4$  part of  $X_8$  in this coupling combines with the  $BF^2$  term required by supersymmetry to give a tree diagram contribution of the form  $F^2R^4$ . And as mentioned above such terms do not appear in hexagon diagrams for abelian gauge factors. Thus, in any theory with an abelian gauge group factor the Green-Schwarz mechanism cannot cancel the anomaly in a way that is simultaneously compatible with supersymmetry. As a result, there are no consistent  $\mathcal{N} = 1$  supergravity theories in ten dimensions with gauge groups having abelian factors. The details of this argument ruling out  $U(1)$  factors were given in a paper with Adams and DeWolfe [16]. This shows, in particular, that the theories with gauge groups  $U(1)^{496}$  and  $E_8 \times U(1)^{248}$ , which were for some time in the “apparent swampland,” cannot be consistent supersymmetric quantum theories of gravity.

Summary: we have shown that there are only four possible consistent ten-dimensional supergravity theories

$$\mathcal{G}^{(10)} = \{\text{IIA, IIB, } SO(32), E_8 \times E_8\}. \quad (31)$$

## 2.5 String theories in ten dimensions

In this section we give a short overview of how quantization of strings and other extended objects can lead to consistent frameworks for describing quantum gravity. For a detailed pedagogical introduction to string theory and branes, the reader should consult Polchinski [7, 8].

The fundamental excitations in any quantum field theory are pointlike quanta associated with localized particles in space-time. While directly quantizing an interacting theory of

gravity in more than three dimensions using quantum field theory methods proves problematic, greater success has been realized in quantizing gravity using extended objects. As we have seen, supersymmetric theories of gravity in higher dimensions often contain antisymmetric  $p$ -form fields. For example, all supergravity theories in ten dimensions contain an antisymmetric two-form field  $B_{\mu\nu}$  in the gravity multiplet, and 11-dimensional supergravity contains an antisymmetric three-form field  $C_{\mu\nu\lambda}$ . These antisymmetric tensor fields can couple to extended charged objects in the same way that the electromagnetic vector potential  $A_\mu$  couples to charged particles. The coupling of a pointlike object with charge  $q$  under a vector field  $A_\mu$  is described by a contribution to the action given by an integral along the particle world-line  $x^\mu(\tau)$

$$S_0 = q \int A_\mu dx^\mu. \quad (32)$$

Similarly, the field  $B_{\mu\nu}$  couples to charged stringlike excitations with one direction of spatial extent through

$$S_1 = \int B_{\mu\nu} dx^\mu dx^\nu, \quad (33)$$

and a 3-form field couples to a membrane with two spatial dimensions in a similar fashion, etc. Just as gravity coupled to electromagnetism in four dimensions admits classical solutions with mass and charge (Reissner-Nordström black holes), gravitational theories in higher dimensions with  $(p+1)$ -form fields admit extended charged “black brane” solutions. In supersymmetric theories, the central charge in the supersymmetry algebra places an upper bound on the charge/mass ratio possible for such brane solutions. Solutions which saturate this bound are known as *extremal* solutions. Extremal brane solutions will be quantized in a quantum theory of gravity. By considering the quantum field theory on the world-volume of the branes, a theory of quantum gravity in the bulk space-time emerges in many cases.

The principle of *brane democracy* [17] states that any of the branes appearing in a theory can in some sense be treated as fundamental degrees of freedom. Indeed, to the extent it is possible to quantize branes in any supergravity theory, the resulting quantum theory seems to give at least a limited description of a consistent quantum gravity theory. The  $p$ -form fields and associated extended objects appearing in 11D and 10D supergravity theories are listed in the following table

<u>Theory</u>	<u>Field</u>	<u>Brane</u>
11D	$C_{\mu\nu\lambda}$	M2-brane (+ dual M5)
$\mathcal{N} = 1, 2$ 10D	$B_{\mu\nu}$	(F) string (+ NS5-brane)
IIA	$A_\mu, C_{\mu\nu\lambda}$	
IIB	$\tilde{B}_{\mu\nu}, D_{\mu\nu\lambda\sigma}^+$ [RR-fields]	{(D)-branes}

For each  $(p+1)$ -form field, there is a corresponding  $p$ -brane that is electrically charged, in the sense that the brane couples to the field in a fashion analogous to (32) and (33). There

is also a dual  $(D - p - 4)$ -brane that couples magnetically, in the sense that there is a flux for the field strength through a space-like  $(p + 2)$ -sphere surrounding the magnetic charge. For example, there is a dual 5-brane charged magnetically under  $B_{\mu\nu}$  in every 10D supergravity theory, so that a 3-sphere surrounding the 5-brane carries nonzero flux  $\int_{S^3} H$ .

The branes that can be quantized in the most well-understood fashion are the “fundamental” strings appearing in all ten-dimensional supergravity theories as objects charged under  $B_{\mu\nu}$ . The perturbative approach to string theory begins by considering the theory on the world-sheet of these strings, described by a sigma model mapping the world-sheet into space-time. This approach leads to five tachyon-free superstring theories in ten dimensions: the type IIA and type IIB theories, whose low-energy limits are the type IIA and type IIB supergravity theories; the heterotic  $SO(32)$  and  $E_8 \times E_8$  theories, with the associated  $\mathcal{N} = 1$  supergravity theories as low-energy limits; and type I string theory, which gives a different description of the  $SO(32)$  theory. These theories give perturbative descriptions of the quantum supergravity theory in each case, and can be directly formulated in terms of a sigma model for backgrounds solving the classical equations of motion with nontrivial geometry, dilaton, and  $B$  fields. The perturbative description is less clear when the background contains nonzero Ramond-Ramond fields, but recent progress has been made in this direction [18, 19].

Just as fundamental strings form a natural route to quantizing ten-dimensional supergravity theories, the membrane carrying charge under the three-form field  $C$  is a natural candidate for quantizing 11-dimensional supergravity. Although a generally covariant quantization of the membrane has not been found, by restricting to light-cone gauge the theory on the membrane can be regulated in such a way that the resulting theory is a simple matrix quantum mechanics theory. This matrix quantum mechanics theory, known as “M(atrix) theory” gives a nonperturbative description of M-theory in light-cone gauge [20, 21, 22].

A major breakthrough in understanding quantum gravity and string theory was made when Polchinski observed that the supergravity brane solutions charged under the R-R fields of type II string theory could be identified with dynamical branes in space-time located on hypersurfaces where open strings end with Dirichlet boundary conditions (“D-branes”). This insight provides two ways of describing the dynamics of such branes: analyzing fluctuations around the classical soliton solution, or quantizing open strings ending on the brane. Quantizing strings connecting  $N$  coincident branes leads to a world-volume  $SU(N)$  gauge theory; in an appropriate near-horizon limit the gauge theory becomes supersymmetric Yang-Mills theory. The celebrated AdS/CFT correspondence is the correspondence between this Yang-Mills theory and the dual gravity theory in the vicinity of the brane [23, 24]. Again, quantizing fluctuations around a brane has led to a theory of gravity. In this case, the gravity theory is in a space of higher dimensionality than the field theory on the brane world-volume, and the correspondence gives a nonperturbative definition of quantum gravity in terms of a field theory. The AdS/CFT correspondence underlies many of the talks at this TASI school. One other situation where quantizing the theory on branes leads to quantum gravity is realized by quantizing the theory on  $N$  D0-branes in type IIA string theory. This gives an alternate route to the M(atrix) model of M-theory mentioned above [21].

The upshot of this discussion is that theories of quantum gravity can be studied by quantizing  $p$ -dimensional extended objects coupled to dynamical  $(p+1)$ -form fields in supergravity theories. String theory, the AdS/CFT correspondence, and M(atrix) theory are all examples of this general principle. In later parts of these lectures we will also use branes as a tool in constructing string vacua in dimensions less than 10.

## 2.6 Summary: string universality in ten dimensions

We showed above that there are only four distinct theories of supergravity in ten dimensions that do not suffer from inconsistencies due to quantum anomalies. Since each of these theories can be realized in string theory, we have string universality for supergravity theories in ten dimensions

$$\mathcal{G}^{(10)} = \mathcal{G}_*^{(10)} = \mathcal{V}_*^{(10)} = \mathcal{V}^{(10)} = \{\text{IIA, IIB, SO}(32), E_8 \times E_8\}. \quad (35)$$

Of course, as mentioned above, we have only considered the discrete field content and symmetries of the theory, and have not proven that each of these quantum theories is perturbatively and nonperturbatively unique. Trying to prove, for example, that all higher-derivative terms in the theory are uniquely determined by supersymmetry and quantum consistency is a further interesting enterprise on which some initial progress has been made [25].

Note that the strategy of analyzing theories in the “apparent swampland”  $\mathcal{G} \setminus \mathcal{V}$  has led to several steps of progress enroute to this result. After Green and Schwarz identified their anomaly cancellation mechanism, the  $E_8 \times E_8$  theory was in the apparent swampland. Identifying a string theory underlying this supergravity theory became a clear challenge, which was met when the “Princeton string quartet” of Gross, Harvey, Martinec and Rohm successfully constructed the  $E_8 \times E_8$  heterotic theory [26]. Thereafter, for some time the theories with  $U(1)^{496}$  and  $E_8 \times U(1)^{248}$  gauge groups remained in the apparent swampland, motivating the eventual demonstration that these theories are inconsistent.

As discussed above, there are actually two string theory realizations of the  $\mathcal{N} = 1$  10D supergravity theory with  $SO(32)$  gauge group, the type I and heterotic  $SO(32)$  string theories. While there is no proof that these theories are equivalent, strong evidence suggests that there is a nonperturbative *duality* symmetry which identifies these two apparently distinct string theories [10, 27]. From the discussion so far, it seems that there are a number of disconnected theories in ten dimensions. These theories are all connected, however, through a network of duality symmetries [28, 10]. In particular, once the theories are compactified to lower dimensions, it becomes clear that apparently different 10D supergravity/superstring theories give rise to different descriptions of the same continuous space of gravity theories in the lower dimension. We now turn to eight dimensions, where we see an explicit example of such a duality symmetry for lower-dimensional theories.

### 3 Supergravity and String Vacua in Eight Dimensions

We begin our consideration of eight-dimensional theories with a brief introduction to eight-dimensional supergravity. We then discuss some general aspects of the compactification of supergravity theories from 11D and 10D to lower dimensions. This sets the stage for introducing two approaches to constructing 8D superstring vacua: heterotic compactifications on a two-torus, and F-theory compactifications on an elliptically fibered K3 surface. We show that these two rather different string constructions give rise to the same set of 8D theories, providing an example of a duality symmetry relating ostensibly very different string constructions.

#### 3.1 Supergravity in eight dimensions

As in 10 dimensions, gravity theories can be constructed in 8 dimensions with either one or two supersymmetries. The  $\mathcal{N} = 2$  8D supergravity theory contains only the supergravity multiplet and thus has a uniquely determined field content. This supergravity theory can be realized through dimensional reduction of 11D or 10D supergravity on a torus, and thus has a natural string theory realization. In each dimension we focus on the supergravity theories with minimal supersymmetry, where novel phenomena emerge. In 8D  $\mathcal{N} = 1$  theories, the new feature present is a wider range of possible gauge groups relative to the highly constrained set of 10D  $\mathcal{N} = 1$  supergravity theories.

Just as in ten dimensions, the minimal  $\mathcal{N} = 1$  supersymmetry in eight dimensions has 16 supercharges, and the multiplets of interest consist of the supergravity and vector multiplets

$$8D \mathcal{N} = 1 \quad \left\{ \begin{array}{lll} \text{SUGRA} & g_{\mu\nu}, B_{\mu\nu}, 2A_\mu, \sigma; \psi_{\mu\alpha}, \chi_\alpha & [48 + 48 \text{ DOF}] \\ \text{vector} & A_\mu, 2\phi, \lambda_\alpha & [8 + 8 \text{ DOF}] \end{array} \right. \quad (36)$$

In eight dimensions, the graviton multiplet contains two vector fields, often referred to as *graviphotons*. Spinors in eight dimensions, as in four dimensions, can be Majorana or Weyl, but not both. Classically, the supergravity multiplet can be coupled to any number of vector multiplets [29, 30]. 8D theories can in principle have pure gauge or mixed gauge-gravitational anomalies from pentagon diagrams [14]. The gauge factor is always of the form  $\text{tr}_R F^3$  or  $\text{tr}_R F^5$  where the trace is taken in the representation  $R$  under which the charged matter transforms; these terms vanish in the adjoint representation, which is the only representation possible for the charged fermions in a supersymmetric theory, so there are no local anomalies in 8D supergravity theories. In our current state of knowledge regarding 8D theories, then, there are no restrictions from anomalies on the gauge group and

$$\mathcal{G}^{(8D, \mathcal{N}=1)} = \mathcal{G} = \{\text{SUGRA} + \text{YM for any } G\}. \quad (37)$$

This is an infinite set of theories with distinct gauge groups. We discuss at the end of this section how this set of theories may be further constrained.

To physicists familiar with supersymmetry in 4 dimensions, or even 10 or 6 dimensions, it may be surprising that the supergravity multiplet in 8 dimensions has a number of degrees

of freedom (96) that is not a power of 2. To understand this it is helpful to briefly review the construction of supermultiplets [31, 8]. In any dimension  $D$ , a massless supermultiplet is formed by starting with a representation of the corresponding little group  $SO(D-2)$  and acting with raising operators associated with the  $\Gamma$  matrices representing the supersymmetry algebra. For example, the  $\mathcal{N} = 1$  supergravity multiplet in 10 dimensions is formed by acting on the 8-dimensional vector representation  $\mathbf{8}_v$  of the little group  $SO(8)$  by the set of supersymmetry raising operators that combine into the  $\mathbf{8}_v + \mathbf{8}'$  representation of  $SO(8)$ . The tensor product of these representations gives

$$\mathbf{8}_v \times (\mathbf{8}_v + \mathbf{8}') = \mathbf{35} + \mathbf{28} + \mathbf{1} + \mathbf{56} + \mathbf{8}' \quad (38)$$

which corresponds to the set of degrees of freedom in the components in the supergravity multiplet. In eight dimensions, the little group is  $SO(6)$ . The 16 supersymmetry raising operators break up into representations  $\mathbf{6} + 2 \times \mathbf{1} + \mathbf{4} + \bar{\mathbf{4}}$  of  $SO(6)$ , and the supergravity multiplet is formed from the massless  $\mathbf{6}$  representation, giving

$$\mathbf{6} \times (\mathbf{6} + 2 \times \mathbf{1} + \mathbf{4} + \bar{\mathbf{4}}) = \mathbf{20} + \mathbf{15} + \mathbf{1} + 2 \times \mathbf{6} + \mathbf{20}_\psi + \bar{\mathbf{20}}_\psi + \mathbf{4} + \bar{\mathbf{4}}, \quad (39)$$

which are the degrees of freedom in the fields of the 8D supergravity multiplet.

### 3.2 Compactification of supergravity theories

We have described above the various supergravity theories in 11 and 10 dimensions. As we have seen, the 10-dimensional type IIA theory can be related to a compactification of the 11D supergravity theory on a circle  $S^1$ . More generally, a variety of supergravity theories in lower dimensions can be constructed by compactifying the 11D and 10D supergravity theories on various geometries. For example, considering a  $D$ -dimensional supergravity theory on a manifold of the form  $X_k \times \mathbb{R}^{1,D-k-1}$ , where  $X_k$  is a  $k$ -dimensional compact manifold, gives rise to a supergravity theory in  $D - k$  dimensions. To have a vacuum solution in the lower-dimensional theory, the compactification must be constructed in such a way that the equations of motion of the  $D$ -dimensional supergravity are satisfied. The massless spectrum in  $D - k$  dimensions then comes from the zero modes of the massless fields in the higher-dimensional theory. If the manifold  $X_k$  is a torus  $T^k$ , then such a compactification amounts to a periodic identification of all fields in the theory (with some choice made for boundary conditions on fermions and higher spin fields). In general such a toroidal compactification preserves all of the supersymmetries of the theory, giving a supergravity theory in the smaller dimension with the same number of supercharges. The spectrum in the toroidally compactified theory has the same number of degrees of freedom as the higher-dimensional theory, organized according to the relativistic symmetry group of the lower-dimensional theory. More complicated geometries can be chosen that break some of the supersymmetry, giving a wider variety of theories in lower dimensions.

In addition to purely geometrical compactifications, additional features such as branes and fluxes can be included that increase the range of possibilities. The inclusion of branes gives constructions such as *intersecting brane models*, where branes wrapping cycles on the

compactification manifold break some of the supersymmetry and give rise to gauge groups and matter. *F-theory* models, as we discuss later in this section, can be thought of as compactifications of the type IIB theory with branes, although they also have a natural interpretation in terms of pure geometry. Including fluxes for the various  $p$ -form fields in the supergravity theories can also break some supersymmetry and increase the variety of possible low-energy theories. We do not discuss compactifications with fluxes much in these lectures. Although they are a rich source of structure in lower-dimensional string vacua, there are no interesting classes of flux compactifications known in dimensions 8 or 6.

For now, we focus on compactifications on purely geometrical spaces, without branes or fluxes. We assume then that space-time is of the form

$$\mathcal{M}_{10} = \mathbb{R}^{1, D-k-1} \times X_k, \quad (40)$$

where  $D = 10$  or  $D = 11$ . We assume that we are expanding the theory around a background where all fields other than the metric (and dilaton when  $D = 10$ ) vanish, with a constant value for the dilaton when  $D = 10$ . To preserve supersymmetry there must be a supersymmetry transformation under which the variation of the gravitino vanishes. Thus, in particular, there must be a spinor  $\eta$  on the space-time so that  $\delta\psi_\mu = D_\mu\eta = 0$ . Such a covariantly constant spinor can exist on a manifold of dimension  $k = 2N$  only when parallel transport on the manifold, or *holonomy*, leaves a component of the spinor unchanged. This is possible when the holonomy around all curves lies in a subgroup  $SU(N)$  of the structure group  $SO(2N)$ . A manifold of dimension  $2N$  has holonomy in  $SU(N)$  if and only if it is a complex Kähler manifold with a nowhere vanishing holomorphic  $N$ -form. Such manifolds are known as *Calabi-Yau* manifolds. We will not need to deal with many technical aspects of Calabi-Yau manifolds in these lectures, but we will learn a few things about some of the simpler Calabi-Yau spaces in the context of various string compactifications.

The simplest example of a Calabi-Yau manifold is a torus of even dimensionality  $T^{2N}$ . For a torus the holonomy is trivial around any curve since the connection vanishes. For compactification on a torus, all supersymmetries of the higher-dimensional theory are retained after compactification. For complex dimension 1, the two-torus is the only Calabi-Yau manifold. For complex dimension 2, besides the torus  $T^4$ , there is one other topological class of Calabi-Yau manifold; this manifold is known as the *K3 surface*. Compactification on a K3 surface breaks half of the supersymmetries of the higher-dimensional theory. We will describe this manifold in more detail and study some of its features below in our discussion of 8-dimensional F-theory vacua. The number of topologically distinct Calabi-Yau manifolds in complex dimension three is much larger; it is not even known if this number is finite. Compactification on a Calabi-Yau threefold reduces the number of supersymmetry generators by a factor of 4.

There are a few other situations in which supersymmetry can be preserved under purely geometrical compactifications. If a 7-manifold has holonomy in the group  $G_2 \subset SO(7)$ , then a supersymmetry is preserved [32]. For example, 11-dimensional supergravity compactified on a  $G_2$ -holonomy 7-fold gives a theory with one supersymmetry in four dimensions, reducing the number of supersymmetry generators by a factor of 8. Similarly, compactification on an

8-fold with holonomy in  $\text{Spin}(7)$  or  $\text{Sp}(2)$  gives theories with supersymmetry reduced by a factor of  $1/16$  or  $3/16$  [33, 34, 35].

A table of the primitive geometric compactifications that admit a covariantly constant spinor, and hence supersymmetry in the dimensionally reduced theory, is given below. Products of these spaces will, of course, also give a supersymmetric dimensionally reduced theory; for example, compactification of a 10D theory on  $K3 \times T^2$  gives a 4D theory with supersymmetry reduced by a factor of  $1/2$ . In this table, CY3, CY4 refer to Calabi-Yau threefolds and fourfolds with holonomy precisely equal to  $SU(N)$ ; product spaces have reduced holonomy and preserve more supersymmetry. For reductions on compact spaces of dimension up to 8, the manifolds in this table and products thereof are the only smooth geometries which preserve some supersymmetry. A more detailed discussion of the mathematics behind this assertion and further references can be found in the book by Gross, Huybrechts and Joyce [36] and in Morrison's TASI notes [34].

(real) dimension	manifold type	holonomy	SUSY
$k$	$T^k$	$\{1\}$	1
4	K3 (CY2)	$SU(2)$	$1/2$
6	CY3	$SU(3)$	$1/4$
7	$G_2$	$G_2$	$1/8$
8	$\text{Sp}(2)$	hyper-Kähler	$3/16$
8	CY4	$SU(4)$	$1/8$
8	$\text{Spin}(7)$	$\text{Spin}(7)$	$1/16$

We can now ask the question: “How can we get a supergravity theory in eight dimensions by a geometric compactification?”. Given what we have learned so far, it would seem that this can only be done by compactifying a 10-dimensional supergravity theory on a two-torus  $T^2$ . Indeed, compactifying IIA and IIB supergravity/string theory on  $T^2$  gives rise to two (dual) descriptions of the same  $\mathcal{N} = 2$  supergravity in eight dimensions, while compactifying  $\mathcal{N} = 1$  supergravity on  $T^2$  gives  $\mathcal{N} = 1$  supergravity in 8D. In the following section we describe the compactification of the heterotic theories to 8D in more detail. We then describe another approach to constructing eight-dimensional supergravity theories through *F-theory*. F-theory can be thought of as a compactification of the type IIB theory on a curved manifold with branes. The branes give rise to an effective 12-dimensional geometric picture, so that  $\mathcal{N} = 1$  supergravity in eight dimensions can be thought of as arising from F-theory on a K3 surface. As we now discuss in more detail, the two approaches of heterotic on  $T^2$  and F-theory on K3 give dual descriptions of the same set of 8D supergravity theories.

$$\begin{array}{ccc}
 \text{het} = \text{I (10D) [16 Q]} & & \text{F-theory (“12D”) [32 Q]} \\
 \downarrow T^2 & \text{dual} & \downarrow \text{K3} \\
 8\text{D [16 Q]} & & 8\text{D [16 Q]}
 \end{array}$$

### 3.3 Heterotic string vacua in eight dimensions

#### 3.3.1 Compactification of 10D $\mathcal{N} = 1$ theories on $T^2$

We now consider the theories that can be realized by compactifying the  $SO(32)$  and  $E_8 \times E_8$  theories from ten dimensions to eight dimensions on a torus  $T^2$ . From the point of view of the 10D  $\mathcal{N} = 1$  supergravity theory, we can deduce the field content of the 8D theory by considering how the representations of the 10D little group  $SO(8)$  for massless fields decompose when reduced to the 8D little group  $SO(6)$ . Operationally, we can perform the reduction of fields by choosing for each 10D Lorentz index  $(0, \dots, 9)$  either an 8D Lorentz index  $(\mu = 0, \dots, 7)$  or an index in one of the compact directions  $(c = 8, 9)$ . For example, the 10D vector field (with massless states transforming in the  $\mathbf{8}_v$  of  $SO(8)$ ) reduces to an 8D vector field  $A_\mu$  (with states in the  $\mathbf{6}$  of  $SO(6)$ ) and two scalar fields  $\phi = A_8, \phi' = A_9$ . Thus, the 8D vector multiplet is simply the dimensional reduction of the 10D vector multiplet. The dimensional reduction of the 10D  $\mathcal{N} = 1$  supergravity multiplet gives the 8D supergravity multiplet and two vector multiplets ( $128 = 96 + 2 \times 16$ ). The fact that the supersymmetry multiplet is reducible after dimensional reduction comes from the decomposition of the  $\mathbf{8}_v$  multiplet on which the SUSY generators act in (38) through  $\mathbf{8}_v \rightarrow \mathbf{6} + 2 \times \mathbf{1}$ . In the anomaly-free 10D  $\mathcal{N} = 1$  supergravity theories, there are 496 gauge bosons. The compactification of the bosonic fields of such a supergravity theory on  $T^2$ , assuming all gauge fields vanish, gives an 8D theory with one supergravity multiplet and 498 vector multiplets.

$$\begin{array}{ccc}
 g_{\mu c}, B_{\mu c}, A_\mu & & \phi, A_c, g_{cc'}, B_{cc'} \\
 \downarrow & & \downarrow \\
 500 A_\mu \quad [U(1)^4 \times G] & & \sigma + 996 \phi \text{ (scalars/moduli)}
 \end{array}$$

The scalar  $\sigma$  in the supergravity multiplet and the two scalars in each vector multiplet combine to give 997 scalar fields in the 8D theory.

From the discussion so far, it would seem that the gauge group of the dimensionally reduced theory must be given by  $G \times U(1)^4$ , where  $G = SO(32)$  or  $E_8 \times E_8$  is the nonabelian gauge group of the 10D theory. In either case, the rank of the 8D gauge group is 20. The gauge group in the reduced theory is generically broken to a smaller group, however, by *Wilson lines* around the circles of the compactification space. From the point of view of the low-energy theory this corresponds to Higgsing the gauge group by turning on expectation values for any of the 992 scalars arising from the dimensional reduction of  $A_8, A_9$ , which transform in the adjoint representation of the gauge group. The gauge group can also be increased to a larger group by a stringy gauge enhancement mechanism. In either case the rank of the group stays fixed at 20. We now consider each of these possibilities in turn.

The possibility of nontrivial Wilson lines for the gauge fields  $A_\mu$  around each of the two directions in the torus can be understood in a straightforward fashion. A constant nonabelian gauge field can be turned on in the directions  $A_8$  and  $A_9$  without breaking supersymmetry, as long as the curvature  $F = [A_8, A_9] = 0$  vanishes. (Note that nonzero curvature gives an energy density  $E \sim F^2$ , which is incompatible with the vanishing vacuum energy required

by supersymmetry in Minkowski space.) In the presence of Wilson loops, the 10D gauge group is broken to  $H \subset G$  where  $H = \{h : [h, A_8] = [h, A_9] = 0\}$ . Since  $A_8, A_9$  can always be chosen as part of a maximal torus, the rank of the resulting 8D gauge group is always 20. This gives a large family of possible unbroken gauge groups for the 8D theory. For generic commuting Wilson lines  $A_8, A_9$ , the nonabelian gauge group will be completely broken to

$$G_{\text{generic}} = U(1)^{20}. \quad (41)$$

In such a generic vacuum, two of the  $U(1)$  factors are graviphotons. Associated with the 16 unbroken generators of the original nonabelian gauge group there are 32 scalar fields. Four additional scalars are associated with the shape and size of the compactification torus. There is one more scalar coming from the 10D dilaton, for a total of 37 scalar fields. These fields parameterize the space of supersymmetric vacua of the theory. There is no potential for these massless scalar fields, which are known as *moduli* of the theory. This gives us a picture of the 8D supergravity theory resulting from the heterotic compactification as having a continuously connected 37-dimensional moduli space of vacua; in lower-dimensional subspaces of the moduli space the gauge group is enhanced.

From the preceding discussion it seems that the maximum enhancement of the gauge group can be to one of the original heterotic groups. As mentioned above, however, string theory also provides a mechanism for enhancing the gauge group further beyond  $U(1)^4 \times SO(32)$  or  $U(1)^4 \times E_8 \times E_8$ . To understand this mechanism, it will be helpful to briefly review some basics of the mathematics of lattices. Lattices are relevant here because the  $k$ -dimensional torus can be thought of as a quotient  $\mathbb{R}^k/\Gamma$ , where  $\Gamma$  is a  $k$ -dimensional lattice. Lattices will also play an important role in many other constructions in these notes.

### 3.3.2 Interlude on lattices

A  $k$ -dimensional lattice is defined to be the subset of  $\mathbb{R}^k$  given by integral linear combinations of a set of  $k$  linearly independent basis vectors  $e_i$

$$\Gamma = \{n_i e_i, n_i \in \mathbb{Z}\}. \quad (42)$$

As in much of the mathematical literature, we are interested in lattices that carry an integral symmetric bilinear inner product

$$v \cdot w \in \mathbb{Z} \quad \forall v, w \in \Gamma. \quad (43)$$

We assume that every lattice discussed in these notes carries such a structure. We present here only some basic aspects of the theory of lattices. For a more thorough introduction to the subject see the text by Conway and Sloane [37].

For a given choice of basis, a lattice  $\Gamma$  can be represented by the integral matrix  $e_i \cdot e_j$ . We will sometimes use  $\Gamma$  to denote this matrix as a convenient shorthand for describing a given lattice. A lattice  $\Gamma$  is *Euclidean* if

$$v \cdot v > 0 \quad \forall v \in \Gamma. \quad (44)$$

This corresponds to the condition that the associated integral matrix is positive definite. More generally, we will be interested in lattices of indefinite signature  $(p, q)$ .

A lattice is said to be *even* if

$$v \cdot v \in 2\mathbb{Z} \quad \forall v \in \Gamma. \quad (45)$$

The *dual*  $\Gamma^*$  of a lattice  $\Gamma$  is defined to be the set of dual vectors whose inner product with all elements of  $\Gamma$  is integral

$$\Gamma^* = \{v : v \cdot w \in \mathbb{Z} \quad \forall w \in \Gamma\}. \quad (46)$$

A lattice is *self-dual* or *unimodular* if it is equal to its dual

$$\Gamma = \Gamma^*. \quad (47)$$

This is equivalent to the condition that the associated matrix has unit determinant,  $\det \Gamma = \pm 1$ . Alternatively, the self-duality condition is equivalent to the condition that the basis vectors  $e_i$  span a unit cell of volume  $\pm 1$ . An important theorem due to Milnor states that any unimodular lattice has signature satisfying

$$p \equiv q \pmod{8} \quad (\Gamma \text{ unimodular}). \quad (48)$$

**Example:** As an example of an even unimodular lattice consider the lattice

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_1^2 = e_2^2 = 0, \quad e_1 \cdot e_2 = 1. \quad (49)$$

This lattice has signature  $(1, 1)$ . A sublattice of  $U$  is spanned by the vectors with respect to which  $U$  is diagonalized

$$(e_1 + e_2)^2 = 2, \quad (e_1 - e_2)^2 = -2, \quad (e_1 + e_2) \cdot (e_1 - e_2) = 0. \quad (50)$$

A particularly noteworthy class of even lattices are the *root lattices* of the algebras associated with the simply-laced Lie groups ( $SU(N)$ ,  $SO(2N)$ ,  $E_6$ ,  $E_7$ ,  $E_8$ ). For such an algebra the simple roots  $r_i$  form a basis for the lattice. The Cartan matrix is formed from the inner products of the simple roots  $a_{ij} = r_i \cdot r_j$ , where the roots are normalized such that  $a_{ii} = 2$ . For simply-laced algebras, all off-diagonal entries in the Cartan matrix are either 0 or -1. These algebras and the associated matrices are often conveniently described in terms of *Dynkin diagrams*. The Dynkin diagram for a given algebra/Cartan matrix is given by drawing a node for each simple root  $r_i$ , and connecting with a line each pair of nodes  $i, j$  satisfying  $r_i \cdot r_j = -1$ . The number of nodes in the Dynkin diagram corresponds to the rank of the associated Lie algebra, which is the same as the dimension of the corresponding lattice. The

lattice and Dynkin diagram associated with the algebra of  $SU(N)$  is denoted  $A_{N-1}$ , while that associated with the algebra of  $SO(2N)$  is denoted  $D_N$ .

**Example:** The lattice  $A_2$  is described by the Cartan matrix associated with the Lie algebra of  $SU(3)$

$$(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (51)$$

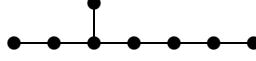
This lattice is even but not unimodular. The associated Dynkin diagram is



**Example:** The lattice  $E_8$  is described by the Cartan matrix associated with the  $E_8$  group

$$(a_{ij}) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad (52)$$

This lattice is both even and unimodular. The associated Dynkin diagram is



As a consequence of (48), a Euclidean lattice can only be self-dual if it has a dimension  $p \equiv 0 \pmod{8}$ .  $E_8$  is the unique Euclidean self-dual lattice of dimension 8. There are two even self-dual Euclidean lattices of dimension 16:

$$E_8 \oplus E_8, \text{ and } \Lambda_{16}, \quad (53)$$

where  $\Lambda_{16}$  is the *Barnes-Wall* lattice. This lattice is closely related to the root lattice  $D_{16}$  of  $SO(32)$ .  $\Lambda_{16}$  includes  $D_{16}$  as a sublattice, and contains an additional set of points corresponding to the weights of a spinor representation of  $SO(32)$  (essentially another copy of  $D_{16}$ , but shifted by an offset moving each lattice point to one of the biggest “holes” in the original lattice).

For lattices of indefinite signature  $(p, q)$ , there is a unique even self-dual lattice with any given signature when  $p \equiv q \pmod{8}$ . This lattice is denoted  $\Gamma^{p,q}$ , and is given by

$$\Gamma^{p,q} = U \oplus \cdots \oplus U \oplus (\pm E_8) \oplus \cdots \oplus (\pm E_8) \quad (54)$$

where the number of factors of  $U$  is  $\min(p, q)$  and the number of factors of  $E_8$  is  $(p - q)/8$ .

### 3.3.3 Toroidal compactification and enhanced symmetries

We can now describe enhanced symmetries of the toroidally compactified heterotic string in terms of the language of lattices. We begin by recalling the quantization of the simple bosonic string on a compact circle of radius  $R$ , following the notation of Polchinski [7]. In terms of world-sheet coordinates  $\sigma, \tau$ , and setting  $\alpha' = 1$ , the position of the string in the compact direction is given by

$$X = x + w\sigma + p\tau + \text{oscillators}(\alpha_n, \tilde{\alpha}_n) \quad (55)$$

$$= x + \frac{1}{\sqrt{2}}l_L(\tau + \sigma) + \frac{1}{\sqrt{2}}l_R(\tau - \sigma) + \dots \quad (56)$$

where the winding number and momenta are quantized through

$$w = mR, \quad p = n/R. \quad (57)$$

T-duality is the symmetry that exchanges  $w \leftrightarrow p$  through  $m \leftrightarrow n$  and  $R \leftrightarrow 1/R$  ( $R \rightarrow \alpha'/R$  when  $\alpha'$  is reinstated). The left- and right-moving momenta  $l_{L,R}$  are related to winding number and momenta through

$$l_L = \frac{p + w}{\sqrt{2}}, \quad l_R = \frac{p - w}{\sqrt{2}}. \quad (58)$$

These momenta live on the even lattice  $U$  from (49), with inner product

$$l \cdot l = l_L^2 - l_R^2 = 2pw = 2nm \in 2\mathbb{Z}. \quad (59)$$

Restoring units, physical momenta are given by

$$k = \sqrt{2/\alpha'} l, \quad (60)$$

and the mass shell condition for string states is

$$M^2 = k_\mu k^\mu = \frac{2}{\alpha'} [l_L^2 + 2(N-1)] = \frac{2}{\alpha'} [l_R^2 + 2(N-1)]. \quad (61)$$

There are then two different classes of massless vector fields arising from closed string states. There are states with both left and right moving oscillator number  $N = \tilde{N} = 1$  and vanishing left and right momenta

$$\alpha_{-1}^\mu \tilde{\alpha}_{-1}^* |l_L = 0; l_R = 0\rangle \pm \alpha_{-1}^* \tilde{\alpha}_{-1}^\mu |l_L = 0; l_R = 0\rangle, \quad (62)$$

where  $*$  denotes the compact direction. These states are the string states associated with the vector bosons  $g^{\mu*}, B^{\mu*}$  in the dimensionally reduced theory. There are additional massless vector states with only one nonzero oscillator number, namely

$$\tilde{\alpha}_{-1}^\mu |l_L; l_R\rangle, \quad l_R = 0, \quad l_L^2 = l \cdot l = 2 \quad (63)$$

and the analogous states with a single  $\alpha_{-1}^\mu$ . These massless states occur in the spectrum precisely when  $R = \sqrt{\alpha'}$ , the self-dual radius for the circle.

The upshot of this analysis is that when there are points in the lattice with  $l \cdot l = 2$  and either  $l_R = 0$  or  $l_L = 0$  then there are additional massless vectors in the theory. In the case of compactification on a single circle  $S^1$ , these massless vectors appear at the self-dual radius, and enhance the generic  $U(1)^2$  gauge symmetry to  $SU(2)^2$ . Note that the condition  $l \cdot l = 2$  is always satisfied by a set of elements of the momentum lattice, while the existence of nonzero lattice vectors satisfying the condition  $l_R = 0$  or  $l_L = 0$  depends upon the values of the scalar moduli — in this case the compactification radius.

Generalizing to a compactification on a higher-dimensional torus  $T^D$ , the momenta  $l$  take values in a signature  $(D, D)$  lattice

$$\Gamma^{D,D} = U \oplus \cdots \oplus U. \quad (64)$$

Again, the generic gauge group arising from compactification is  $U(1)^{2D}$ , but enhanced symmetries arise for states with  $l \cdot l = 2$  and either  $l_R = 0$  or  $l_L = 0$ .

The moduli space of theories associated with compactification on  $T^D$  can be parameterized by the embeddings of the lattice  $\Gamma^{D,D}$  in the momentum space  $l_L, l_R$  of the compact directions with inner product  $l_L^2 - l_R^2$ . Given one embedding of the lattice in these coordinates, any other embedding can be realized by acting with a transformation in the group  $SO(D, D)$  that preserves the inner product. Two embeddings that are related by a transformation in  $SO(D) \times SO(D)$  that separately preserves  $l_L^2$  and  $l_R^2$  are physically equivalent, so the moduli space is locally given by  $SO(D, D)/SO(D) \times SO(D)$ . There is a further equivalence between embeddings related by a transformation in  $G_D = SO(D, D; \mathbb{Z})$ , the discrete (T-duality) group of transformations that map the lattice  $\Gamma^{D,D}$  to itself. The global space of toroidal compactifications is thus given by the Narain moduli space

$$SO(D, D; \mathbb{Z}) \backslash SO(D, D)/SO(D) \times SO(D), \quad (65)$$

where the discrete duality group acts on the left, independently from the  $SO(D) \times SO(D)$  symmetry acting on the right. As a simple example, for the type IIA and IIB theories compactified on a single circle ( $D = 1$ ), the T-duality group is the  $\mathbb{Z}_2$  group whose nontrivial element exchanges momentum and winding, giving  $n \leftrightarrow m$  in (59). This T-duality transformation gives a duality symmetry relating type IIA string theory compactified on a circle of radius  $R$  and the IIB theory compactified on a circle of radius  $\alpha'/R$ .

For the toroidal compactifications just discussed, the unimodular form of the lattice (64) follows automatically from the construction. More generally, however, we can consider a world-sheet theory with  $p$  chiral left-moving bosons  $\phi^a$  and  $q$  chiral right-moving bosons  $\tilde{\phi}^b$ . The operators in such a theory are of the form

$$\mathcal{O}_{l, \tilde{l}} = \exp \left( i l_a \phi^a + i \tilde{l}_b \tilde{\phi}^b \right). \quad (66)$$

The closure of the set of operators under operator products implies that  $(l, \tilde{l})$  lie in a lattice  $\Gamma$  of dimension  $p + q$ . The single-valued nature of the operator product implies that

$$l_a l_a - \tilde{l}_b \tilde{l}_b \in \mathbb{Z}, \quad (67)$$

so that the lattice is integral and has signature  $(p, q)$ . For a consistent string theory, the world-volume theory must be modular invariant. As shown by Narain [38, 39] (see also Polchinski [7] for more details), modular invariance of the world-volume theory implies that the lattice  $\Gamma$  must be even and self-dual. For  $p = q$ , then, the lattice (64) arising from toroidal compactification is the only lattice possible for a consistent string theory.

### 3.3.4 Lattices and compactification of the heterotic string

We can apply the preceding general discussion to the heterotic string. The world-sheet degrees of freedom on the heterotic string are 10 chiral left-moving bosons and 26 chiral right-moving bosons. (Note that we use the opposite convention from Polchinski for the right-left splitting of degrees of freedom to match the dominant negative signature convention for the associated lattices.) In ten dimensions, ten of the right-moving bosons correspond to space-time momenta, so the compactification lattice is an even unimodular lattice  $\Gamma^{0,16}$ . From (53) we know that there are only two possibilities for 16-dimensional Euclidean even unimodular lattices,

$$(-)\Gamma^{0,16} = E_8 \oplus E_8, \text{ or } (-)\Gamma^{0,16} = \Lambda_{16}. \quad (68)$$

These are precisely the heterotic theories giving  $\mathcal{N} = 1$  10D supergravity with the gauge groups  $E_8 \times E_8$  and  $SO(32)$ . Since  $\Lambda_{16}$  is not exactly the root lattice of  $SO(32)$ , the correct symmetry group for the latter theory is actually  $\text{Spin}(32)/\mathbb{Z}_2$ , though it is generally referred to as the “ $SO(32)$ ” theory. The group  $\text{Spin}(32)$  is the simply connected cover of  $SO(32)$  just as  $SU(2)$  is the simply connected cover of  $SO(3)$ .  $\text{Spin}(32)$  has a center  $\mathbb{Z}_2^2$ , and the group  $SO(32)$  is given by the quotient  $SO(32) = \text{Spin}(32)/\mathbb{Z}_2$  by a particular  $\mathbb{Z}_2$  subgroup of the center. The quotients by the other two  $\mathbb{Z}_2$  factors both give the same group, which is different from  $SO(32)$  and generally denoted  $\text{Spin}(32)/\mathbb{Z}_2$ . Further details related to the discrete quotient are discussed in the references [6, 40].

Now let us consider compactification of the heterotic string to dimensions below 10. Consider first the compactification on  $S^1$  to 9 dimensions. In this case, one of the left-moving bosons and 17 of the right-moving bosons correspond to internal degrees of freedom associated with the compactification, and the resulting lattice is

$$\Gamma^{1,17} = U \oplus (-E_8) \oplus (-E_8). \quad (69)$$

Note that this lattice is the unique even unimodular lattice of signature  $(1, 17)$ . It follows that compactification of both the  $E_8 \times E_8$  and  $\text{Spin}(32)/\mathbb{Z}_2$  theories on a circle  $S^1$  give the same moduli space of theories in 9 dimensions.

Compactifying to eight dimensions, the lattice is

$$\Gamma^{2,18} = U \oplus U \oplus (-E_8) \oplus (-E_8). \quad (70)$$

The moduli space for this theory is the 36-dimensional space

$$SO(2, 18; \mathbb{Z}) \backslash SO(2, 18; \mathbb{R}) / SO(2) \times SO(18), \quad (71)$$

along with the single scalar in the gravity multiplet (the dilaton). The generic gauge group is  $U(1)^{20}$ , as discussed above. It is possible to generate an enhanced gauge group by choosing moduli where there are additional massless gauge bosons. This can produce gauge groups even larger than the original  $E_8 \times E_8$  or  $SO(32)$  of the uncompactified theory. In the heterotic string the ground state for the left-moving oscillators is massless, like the type II string, and only the ground state for right-moving oscillators is tachyonic. So enhanced gauge symmetries only arise when  $l_R^2 = 2, l_L^2 = 0$ . Since there are 18 right-moving bosons, the symmetry group is determined by the sublattice of  $\Gamma^{2,18}$  lying in the negative-signature 18-dimensional space spanned by the  $l_R$ . The possible enhanced nonabelian gauge symmetry groups of the theory are therefore precisely the set of groups whose root lattices  $G$  admit an embedding into  $\Gamma^{2,18}$ .

$$G : -\Lambda_G \hookrightarrow \Gamma^{2,18}. \quad (72)$$

Such groups can have rank up to 18. For example, there is an embedding

$$-D_{18} \hookrightarrow \Gamma^{2,18} \quad (73)$$

from which it follows that the group  $SO(36)$  can be realized in the 8D heterotic theory [8]. Similarly, it was shown by Ganor, Morrison and Seiberg [41] that  $-A_{17}$  can be embedded in  $\Gamma^{2,18}$ , associated with a theory having gauge group  $SU(18)/\mathbb{Z}_3$ .

Determining which groups do or do not admit an embedding of the form (72) is a non-trivial problem. Fortunately, some powerful theorems on lattice embeddings were proven by Nikulin [42] that state necessary and sufficient conditions for such an embedding to be possible. In many cases it is also possible to prove that the embedding, when it exists, is unique up to lattice automorphisms. The precise statement of these theorems is somewhat intricate; in many situations, however, the following simplified theorem, which is a corollary of the stronger theorems, is sufficient.

### Theorem [Nikulin]

Let  $S$  be an even lattice of signature  $(s_+, s_-)$  and let  $T$  be an even, unimodular lattice of signature  $(t_+, t_-)$ . There exists an embedding of  $S$  into  $T$  provided the following conditions hold:

1.  $t_+ \geq s_+$  and  $t_- \geq s_-$
2.  $t_+ + t_- - s_+ - s_- > l(S^*/S)$

where the lattice quotient  $S^*/S$  is a finite abelian group, and  $l(S^*/S)$  is the minimum number of generators of this group  $\square$

It follows from this theorem, for example, that the rank 18 group  $SU(19)$  can be realized as the gauge group of a heterotic string compactification in 8D. The associated lattice  $S = -A_{18}$  has  $s_+ = 0, s_- = 18$ , so satisfies the first condition for  $T = \Gamma^{2,18}$ . For  $A_{n-1}$  the discrete group  $S^*/S$  is  $\mathbb{Z}_n$  [37] (an easy exercise for the reader is to check this for  $A_2$ ). Thus, for  $S = A_{18}$ ,  $S^*/S = \mathbb{Z}_{19}$ , which is a cyclic group with one generator, so the second condition becomes  $20 - 18 > 1$  and is satisfied. The structure of the discrete group  $S^*/S$  also affects

the global structure of the gauge group, for example leading to the  $\mathbb{Z}_3$  quotient in the case of  $A_{17}$  mentioned above [41].

The stronger theorems proven by Nikulin [42] use more detailed aspects of the number-theoretic structure of  $S^*/S$ . The proofs are carried out using  $p$ -adic analysis.

This concludes our brief introduction to heterotic compactifications to 8D.

### 3.4 F-theory vacua in eight dimensions

We now turn to another approach to string compactifications, known as *F-theory* [43, 44, 45]. F-theory can be thought of in several ways: as a limit of a class of type IIA or M-theory vacua, or as a framework for characterizing nonperturbative type IIB vacua<sup>3</sup>. We will primarily approach the subject from the latter perspective in these lectures. Other pedagogical introductions to F-theory can be found in the Les Houches notes of Denef [46] and in the TASI notes by Morrison [34].

#### 3.4.1 7-branes and geometry

As described in Section 2, type IIB supergravity has a classical  $SL(2, \mathbb{R})$  symmetry. In the quantum theory, the spectrum of string excitations charged under the  $B$  fields is quantized, and this symmetry is broken to  $SL(2, \mathbb{Z})$ . Just as the two-form fields  $B, \tilde{B}$  transform as a doublet under this symmetry, fundamental strings and Dirichlet strings (D1-branes) also transform as a doublet under this discrete group. The group  $SL(2, \mathbb{Z})$  thus gives a group of nonperturbative duality symmetries for the IIB string. The axion and dilaton combine into a complex *axiodilaton*

$$\tau = \chi + ie^{-\phi}. \quad (74)$$

The imaginary part of the axiodilaton is positive, so  $\tau$  lives in the upper half-plane of  $\mathbb{C}$ . The axiodilaton transforms under  $SL(2, \mathbb{Z})$  as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1. \quad (75)$$

This group is generated by the transformations

$$T : \quad \tau \rightarrow \tau + 1 \quad (76)$$

$$S : \quad \tau \rightarrow -1/\tau \quad (77)$$

and is precisely the group of transformations on the upper half-plane corresponding to modular transformations on a two-torus parameterized by the complex structure parameter  $\tau$ . It is thus natural to interpret the set of possible values for the axiodilaton as the modular parameter for a two-torus. In the language of algebraic geometry, a two-torus equipped with a complex structure is an *elliptic curve*.

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<sup>3</sup>Most of what I know about F-theory I learned from David Morrison, my guru in the way of F-theory. Credit for any clear and correct insights in the F-theory portions of these notes should go to him, while I of course am responsible for all errors and confusion.

A D7-brane in the IIB theory is magnetically charged under the axiodilaton. This means that the value of the axion field undergoes a monodromy around a small circle surrounding the D7-brane in  $\mathbb{R}^9$ , given by the  $T$  modular transformation

$$\tau \rightarrow \tau + 1. \quad (78)$$

Acting on a D7-brane with an  $SL(2, \mathbb{Z})$  transformation produces a more general class of  $(p, q)$  7-branes, with monodromies conjugate to (78). A supergravity background containing a configuration of  $(p, q)$  7-branes can be geometrized by interpreting  $\tau$  at each point in space-time as parameterizing a torus locally fibered over space-time. This leads to the essential geometric picture of F-theory, a space that is *elliptically fibered* over space-time. While this geometric picture suggests a 12-dimensional enhancement of space-time, it is important to emphasize that there is no underlying 12-dimensional supergravity theory associated with this picture. The presence of different kinds of 7-branes can produce an axiodilaton configuration that may be strongly coupled in any duality frame, and which is therefore essentially nonperturbative. While F-theory captures many nonperturbative aspects of the moduli space of supersymmetric string vacua, there are also limitations to this approach as it is currently understood. F-theory does not at this time have an intrinsic definition including a dynamical principle. Thus, explicit computations, for example of the metric on the compact space or the effect of internal fluxes, depend upon finding a dual perturbative description in which the desired physics can be formulated and computed more precisely.

An alternative to the type IIB picture of F-theory is through a limit of compactification of M-theory. Consider a compactification of the 11-dimensional M-theory on a torus with modular parameter  $\tau$ . Treating one cycle of the torus as compactification of M-theory to type IIA, we have the type IIA theory on a circle of radius  $R$  parameterizing the other cycle of the original torus. By T-duality, as discussed in Section 3.3.3, this is equivalent to type IIB on a circle of radius  $\alpha'/R$ . In the limit where the size of the original torus becomes small,  $R \rightarrow 0$  and this becomes the type IIB theory in an infinite flat space with axiodilaton parameter  $\tau$ . In principle, this construction can be fibered over any base space, reproducing F-theory as a limit of M-theory compactified on a torus shrunk to 0 size; for a more detailed pedagogical description of this picture of F-theory, see Denef's Les Houches notes [46].

An F-theory compactification is associated with an elliptic fibration over a compact base space  $B$  of complex dimension  $d$ , giving a space-time theory in dimension  $D = 10 - 2d$ . For the theory to be supersymmetric, the total space of the elliptic fibration must be a Calabi-Yau manifold. Thus, for example, F-theory vacua in eight dimensions are described by an elliptically fibered Calabi-Yau manifold with two complex dimensions, F-theory vacua in 6 dimensions come from elliptically fibered Calabi-Yau threefolds, and F-theory vacua in 4 dimensions come from elliptically fibered Calabi-Yau fourfolds. We denote the total space of the elliptic fibration by  $X$ , characterizing the fibration through the following diagram

$$\begin{array}{ccc} T^2 & \longrightarrow & X \\ & & \downarrow \\ & & B \end{array}$$

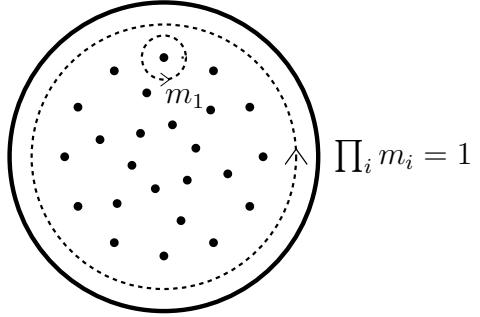


Figure 4: K3 as an elliptic fibration over the sphere  $S^2$ . The fibration is singular at 24 points. The product of the monodromies around the individual points must be the identity, as the associated curve is contractable.

Away from 7-brane sources, the axiodilaton varies smoothly, describing a smooth elliptic fibration over the base space. At the position of a 7-brane, the elliptic fibration becomes singular. This does not mean, however, that the total space  $X$  is necessarily singular. As a rough analogy, consider the two-sphere  $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$  as a circle fibration over a line segment parameterized by  $z \in [0, 1]$ . At every  $z \neq 0, 1$  there is a copy of the circle that varies smoothly with  $z$ . The fibration is singular at the endpoints, where the circle degenerates to a point. But the total space  $S^2$  is still smooth. A similar thing happens in the vicinity of a  $(p, q)$  7-brane in F-theory. The total space  $X$  is smooth, though the structure of the fibration becomes singular.

We now focus specifically on compactifications of F-theory to 8 dimensions, the simplest class of F-theory vacua. First we discuss the structure of the base  $B$  and elliptically fibered Calabi-Yau complex two-fold  $X$ . Like any codimension two massive object, 7-branes produce localized curvature in space-time that leads to a deficit angle in the geometry away from the brane. For a single 7-brane, the metric at large distances becomes asymptotically flat, but the total angle around the brane is reduced by  $\pi/6$ . This can be seen from the ratio between the circumference and diameter of a large circle, which approaches  $11\pi/6$  (instead of  $2\pi$ ) as the circle becomes large. For F-theory to be defined on a compact space, we need enough 7-branes to close the transverse 2D space into a compact geometry. With 12 7-branes, the deficit angle becomes  $2\pi$ , so that the asymptotic space becomes a cylinder (no change in circumference at large radius). Putting together two such configurations, we can construct a topological sphere from a space-time containing 24 7-branes. This can also be understood from the Gauss-Bonnet theorem, which states that the total curvature of a sphere is  $4\pi$ , where each 7-brane locally contributes  $\pi/6$  to the curvature.

Thus, a compactification of F-theory to eight dimensions is associated with a collection of 24 7-branes on a base space that is topologically a two-sphere  $B = S^2$ . Around each 7-brane there is a monodromy  $m_i$  acting on  $\tau$ . The product of the monodromies must be trivial,  $\prod_i m_i = 1$ , since a curve surrounding all 24 7-branes can be contracted to a point on the sphere (see Figure 4). Note that not all the 7-branes can be D7-branes, since if each  $m_i$  gives the modular transformation (78) then  $\prod_i m_i$  would give the modular transformation  $T^{24} : \tau \rightarrow \tau + 24 \neq \tau$ . The total space  $X$  of the elliptic fibration must be a Calabi-Yau two-

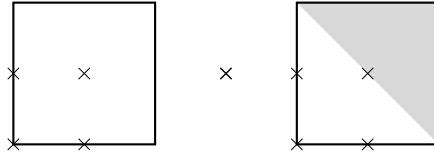


Figure 5: An orbifold limit of the K3 surface can be viewed as the space  $T^4/\mathbb{Z}_2$ , which is locally flat except at 16 orbifold points (marked with “x”’s). The product of the regions on the two 2-tori that have not been marked in gray gives a fundamental domain for the orbifold space.

fold, and is not a torus if there is only one supersymmetry in 8 dimensions.  $X$  is therefore a K3 surface. While the presentation of K3 as an elliptic fibration over  $S^2$  is the principal characterization of K3 that we will use in studying F-theory, it is useful to briefly digress on the structure of the K3 surface as seen from other perspectives.

### 3.4.2 The K3 surface

We now introduce some basic features of the K3 surface. For a much more detailed introduction to the mathematics and physical applications of K3 surfaces, see the lecture notes by Aspinwall on K3 [47].

The K3 surface is the only topological type of Calabi-Yau manifold of complex dimension two besides the four-torus  $T^4$ , as mentioned above. K3 surfaces are simply connected, so they have no first homotopy ( $\pi_1$ ) or homology ( $H_1$ ) structure. K3 surfaces have a 22-dimensional second homology group  $H_2(K3, \mathbb{Z})$ . The intersection pairing on this homology group gives the lattice  $\Gamma^{3,19}$ , which as discussed above must take the form

$$H_2(K3, \mathbb{Z}) = \Gamma^{3,19} = U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8). \quad (79)$$

One of the simplest ways to view the K3 surface is in a particular limit where all the curvature is concentrated at points and the surface can be viewed as an orbifold of a torus. Consider the torus  $T^4$ , considered as a product of two 2-tori with modular parameters  $\tau_1 = \tau_2 = i$ , so that coordinates on the two 2-tori are subject to the usual identifications on each 2-torus  $z_i \sim z_i + 1 \sim z_i + \tau$ . We now take an orbifold of the space by imposing the additional identification of points related through the  $\mathbb{Z}_2$  action

$$\rho : (z_1, z_2) \rightarrow (-z_1, -z_2). \quad (80)$$

The resulting space is flat everywhere except at 16 orbifold points that are locally of the form  $\mathbb{C}^2/\mathbb{Z}_2$  (see Figure 5). This singular limit of the K3 surface can be continuously deformed into a smooth K3 surface by *blowing up* the 16 singular points. The blowing-up procedure, which will be described in more detail below, essentially involves replacing a singular point in a surface with a complex projective space  $\mathbb{CP}^1$  consisting of the set of limiting lines approaching the singular point (note that all projective spaces discussed in these lectures are complex

projective spaces; from here on we denote  $\mathbb{P}^n$  for  $\mathbb{CP}^n$ ). For a singularity of the form  $\mathbb{C}^2/\mathbb{Z}_2$ , blowing up the singular point at the origin produces a smooth space. From this picture we can see fairly clearly how the 22 homology cycles arise. There are six homology classes  $\bar{\pi}_{ij}$  in the original  $T^4$  corresponding to cycles wrapped around the  $ij$  coordinate axes (where the real coordinate axes are labeled by  $i = 1, 2, 3, 4$ ). In the  $T^4$  these have intersection numbers

$$\bar{\pi}_{ij} \circ \bar{\pi}_{kl} = -\epsilon_{ijkl}. \quad (81)$$

In the orbifold space, there are closely related cycles  $\pi_{ij}$ . These lift in the covering space to two copies of  $\bar{\pi}_{ij}$ , with 4 intersections in the covering space. The intersection in the orbifold space is then

$$\pi_{ij} \circ \pi_{kl} = -2 \epsilon_{ijkl}. \quad (82)$$

Another 16 cycles come from the  $\mathbb{P}^1$ 's formed by blowing up the 16 singular points. We denote these cycles  $e_{ij}$ , where the values of the indices  $i, j \in \{1, \dots, 4\}$  correspond to the points  $(0, 0), (1/2, 0), (0, \tau/2), (1/2, \tau/2)$  on the two toroidal factors  $T^2$ . The inner product on these cycles is  $e_{ik} \cdot e_{jl} = -2 \delta_{ij} \delta_{kl}$ . The 22 cycles  $\pi_{ij}, e_{ij}$  span a 22-dimensional lattice. This is not quite the complete homology lattice of K3, however. This is a sublattice, known as the Kummer lattice, of the full homology lattice. To complete the full lattice additional fractional cycles must be added, such as

$$\gamma_{13} = \frac{1}{2} \pi_{13} + \frac{1}{2} (e_{11} + e_{12} + e_{21} + e_{22}). \quad (83)$$

These cycles are topologically spheres, which can be seen from  $\gamma_{13} \cdot \gamma_{13} = -2 = 2g - 2$  where  $g = 0$  is the genus of the surface<sup>4</sup>. Including a set of such fractional cycles gives a complete set of generators for the homology lattice (79) of K3, as reviewed in more detail in several references [47, 48, 49].

### 3.4.3 Elliptically fibered K3 surfaces

Now we return to our discussion of F-theory. As described above, F-theory on a K3 surface is described geometrically by an elliptic fibration over the sphere  $S^2$  with 24 singularities, where each singularity has a monodromy conjugate to  $\tau \rightarrow \tau + 1$ . Locally, an elliptic fibration can be described by a modular parameter  $\tau$  varying holomorphically over the base  $\mathbb{P}^1$ . Generically, the total space of such a fibration is a smooth K3 surface.

A convenient algebraic-geometric description of an elliptic curve is as the set of points in the projective plane satisfying a cubic equation. More specifically, consider the projective space  $\mathbb{P}^{2,3,1}$  defined by the set of coordinates  $(x, y, z)$  with the equivalence

$$(x, y, z) \sim (\lambda^2 x, \lambda^3 y, \lambda z), \quad \forall \lambda \in \mathbb{C} \setminus \{0\}. \quad (84)$$

The equation

$$F = -y^2 + x^3 + f x z^4 + g z^6 = 0 \quad (85)$$

---

<sup>4</sup>This formula relating the intersection form of a curve with itself to the genus of the curve is the special case of a more general formula (91), which we derive later, for a curve in flat or toroidal space.

defines a complex curve of genus one, *i.e.*, an elliptic curve, for general complex values of  $f$  and  $g$ . We can take a local coordinate chart leaving out the points where  $z = 0$  by using eq. (84) to set  $z = 1$  so that the curve in the local  $(x, y)$  coordinate chart is defined by

$$F = -y^2 + x^3 + fx + g = 0. \quad (86)$$

This is known as the *Weierstrass* form of the elliptic curve. One easy way to see that this equation should define a genus one curve is to think of  $x$  as giving a local coordinate chart on  $\mathbb{P}^1$ . The vanishing of  $F$  then gives  $y$  as a double-valued function of  $x$ , with branch points at the points where the cubic  $x^3 + fx + g$  vanishes. Generically there are 3 such branch points, and another branch point at  $x = \infty$  ( $z = 0$ ), so (85) describes a branched cover of  $\mathbb{P}^1$  with 4 branch points, which is topologically a torus or genus one curve. Note that while there are two complex parameters  $f, g$  defining the Weierstrass model, the moduli space of elliptic curves is only one-dimensional; rescaling  $f \rightarrow \lambda^4 f, g \rightarrow \lambda^6 g, x \rightarrow \lambda^2 x, y \rightarrow \lambda^3 y$  gives equivalent elliptic curves.

We can now give a more precise mathematical description of a K3 surface realized as an elliptic fibration over a complex one-dimensional base space such as  $B = \mathbb{P}^1$ . Consider  $w$  as a local coordinate on  $B$ . An elliptic fibration is given locally by a choice of elliptic curve at each point on  $B$ . We can parameterize such a family of elliptic curves in the Weierstrass form by taking  $f, g$  to be functions of  $w$

$$F = -y^2 + x^3 + f(w)x + g(w) = 0. \quad (87)$$

This gives a local Weierstrass description of the elliptic fibration as a hypersurface in  $\mathbb{C}^3$ . To give the global description of the surface, we can reinstate the projective variable  $z$  in the fiber, and a projective variable  $v$  in the base. The Weierstrass form for an elliptic fibration selects a point at  $\infty$  on each elliptic curve (the point at  $z = 0$ ). This choice of a point on each fiber defines a global section of the fibration. Thus, F-theory is defined by a compactification on an elliptically fibered Calabi-Yau *with section*.

A (complex) codimension one space defined as the vanishing locus in  $\mathbb{C}^n$  of a single function such as (86) is singular when all derivatives of the function simultaneously vanish. Such singularities are treated rigorously in the mathematical framework of algebraic geometry, but we can understand this simply by noting that level surfaces of the function define locally smooth complex analytic (and algebraic) sets as long as the gradient of the function is nonvanishing. Thus, the elliptic curve defined by (86) is singular at a point  $(x, y)$  when

$$F = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial x} = -2y = x^3 + fx + g = 3x^2 + f = 0, \quad (88)$$

Combining these equations, we have  $3x^3 + 3fx + 3g = 2fx + 3g = 0 \Rightarrow x = -3g/2f$ , so the curve is singular when the *discriminant*  $\Delta$  vanishes

$$\Delta = 4f^3 + 27g^2 = 0. \quad (89)$$

Considering again the elliptic fibration over the base  $\mathbb{P}^1$ , we take  $f(w)$  and  $g(w)$  to be functions on the base, and the K3 surface is locally defined by (87). At a singularity where the

discriminant (89) vanishes the partial derivative of  $F$  with respect to  $w$ ,  $\partial F/\partial w = f'x + g'$ , is generically nonzero, so at such points the total space of the K3 surface is locally nonzero even though the fiber degenerates. We generically expect 24 singularities in the elliptic fibration as discussed above. Thus, the discriminant should be a degree 24 polynomial in the coordinate  $w$  on the base. We therefore expect that  $f(w)$  has degree 8 and  $g(w)$  has degree 12.

It is helpful to view  $f, g$ , and  $\Delta$  from a more global perspective. For this we need a little more mathematical machinery, which will be extremely helpful in understanding F-theory compactifications on Calabi-Yau manifolds of higher dimension. While these notes are intended to be relatively self-contained, the reader interested in understanding the concepts outlined in the next few paragraphs more thoroughly may find it useful to study some basic aspects of algebraic geometry. For the material presented here, the text of Perrin [50] is a good starting point. The book by Barth, Hulek, Peters, and Van de Ven [51] also contains a great deal of useful material on compact complex surfaces, including aspects of the structure of elliptic fibrations.

While locally  $f$  and  $g$  are functions of the coordinate  $w$  taking values in  $\mathbb{C}$ , when these functions are considered on the complete base  $B = \mathbb{P}^1$  they are actually sections of *line bundles* over  $\mathbb{P}^1$ . Complex line bundles over any complex space  $X$  are characterized by their first Chern class  $c_1(X)$ , and are thus associated with an element of  $H^2(X, \mathbb{Z})$ . In particular, for  $X = \mathbb{P}^1$ , the first Chern class is in  $H^2(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}$ .

We will freely move between the cohomology representative of a line bundle on a space  $X$  and the Poincaré dual homology class, which is a *divisor* on  $X$ . A divisor on  $X$  is a linear combination (over  $\mathbb{Z}$ ) of irreducible algebraic hypersurfaces of the form  $\sum_i n_i H_i$ . There is a one-to-one correspondence between divisors and line bundles over any complex space  $X$ . The divisor associated with a line bundle  $\mathcal{L}$  can be thought of as the linear combination of the homology class containing the zero locus of a section of  $\mathcal{L}$  and the negative of the homology class containing the poles of the section. We will refer to the line bundle associated with a divisor  $D$  as  $\mathcal{O}(D)$ . On  $\mathbb{P}^1$  the only irreducible algebraic hypersurface is (the homology class of) a point  $P$ , so every divisor is characterized as a multiple  $nP$ . For example, the function  $z$  on  $\mathbb{P}^1$  can be extended to a section of the line bundle  $T\mathbb{P}^1 = \mathcal{O}(2P)$ , and vanishes at two points (0 and  $z = \infty \rightarrow w = 1/z = 0$ ). On the other hand, a section  $dz$  of the cotangent bundle goes to  $dz = -dw/w^2$  on the chart around  $z = 1/w = \infty$ , and has a double pole at  $w = 0$ , so  $T^*\mathbb{P}^1 = \mathcal{O}(-2P)$ . We can take the product of two line bundles by multiplying the transition functions; this corresponds to adding the corresponding divisors  $\mathcal{O}(D) \otimes \mathcal{O}(F) = \mathcal{O}(D + F)$ .

On any complex manifold there is a special class known as the *canonical class*,  $K$ . On a complex manifold of dimension  $d$ , the canonical class corresponds to the line bundle associated with the  $d$ th power of the cotangent bundle, and is given locally by the maximum antisymmetric power of the holomorphic differential,  $dz_1 \wedge \cdots \wedge dz_d$ . The canonical class is essentially a measure of the total curvature of a space. On  $\mathbb{P}^1$  the canonical class is

$$K = c_1(T^*) = -2P. \quad (90)$$

A manifold in any complex dimension is Calabi-Yau if and only if<sup>5</sup> the manifold is Kähler and  $K = 0$ .

Given the machinery just defined, we can now understand a formula that will be useful in studying several relevant aspects of curves on surfaces. For a smooth curve  $C$  on a surface  $S$  with canonical class  $K_S$ , we have

$$(K_S + C) \cdot C = 2g - 2. \quad (91)$$

This statement follows from a fairly straightforward three-line argument that uses some important theorems from elementary algebraic geometry. The first step is the *adjunction formula*, which determines the canonical class on  $C$  in terms of  $K_S$ ,  $\mathcal{O}(K_S) \otimes \mathcal{O}(C)|_C = \mathcal{O}(K_C)$ , where  $|$  denotes the restriction of the line bundle to  $C$ . The second step is relating this restriction to the intersection form using  $\mathcal{O}(D)|_C = D \cdot C$ . And the third step uses the *Riemann-Roch* theorem, which essentially says that on a smooth curve  $C$ ,  $\mathcal{O}(K_C) = 2g - 2$ . In flat space, or on a torus or K3 surface,  $K_S = 0$  so (91) becomes  $C \cdot C = 2g - 2$ , which we used in discussing the K3 surface earlier.

As another simple application of the adjunction formula, consider a hypersurface  $D$  in projective space  $\mathbb{P}^{d+1}$  defined by a degree  $d + 2$  homogeneous polynomial. There is a single irreducible hypersurface  $H$  (up to linear equivalence) on  $\mathbb{P}^{d+1}$ , associated with the vanishing of any coordinate function  $z_i$ . The canonical class of  $\mathbb{P}^{d+1}$  is  $K = \mathcal{O}(-(d + 2)H)$ , while the divisor  $D$  associated with a degree  $d + 2$  polynomial is  $D = (d + 2)H$ , so the canonical class on  $D$  is  $\mathcal{O}(K_D) = \mathcal{O}(K + D)|_D = 0$ . Thus, any such  $D$  is a Calabi-Yau. This gives an alternative proof that the cubic on  $\mathbb{P}^2$  gives an elliptic curve. Similarly, a quartic on  $\mathbb{P}^3$  gives a K3 surface, and a quintic on  $\mathbb{P}^4$  gives a Calabi-Yau threefold.

Returning to our elliptically fibered K3 surface, in terms of the canonical class,  $f$  and  $g$  are sections of the line bundles  $\mathcal{O}(-4K)$  and  $\mathcal{O}(-6K)$ , and  $\Delta$  is a section of the bundle  $\mathcal{O}(-12K)$ . Characterizing  $\Delta$  by the curvature class of the associated bundle, the condition for an elliptic fibration to describe an elliptically fibered Calabi-Yau surface over  $B$  can be written as

$$-12K = \Delta. \quad (92)$$

We will refer to this condition as the “Kodaira condition”. It is a special case of the relation proven by Kodaira between the canonical class of the total space of an elliptic fibration and the canonical class of the base. In the case  $B = \mathbb{P}^1$ , the Kodaira condition states that  $\Delta = 24P$ , which is just the statement above that the singularity locus consists of 24 points on the sphere  $S^2$ .

The space of elliptically fibered K3 surfaces is parameterized by the coefficients of the polynomials (sections)  $f(w), g(w)$ . The coefficients are moduli that parameterize the complex structure of the K3 surface just as  $\tau$  parameterizes the complex structure of  $T^2$ . The number of coefficients is  $9 + 13 = 22$ . This parameterization is, however, redundant. Just as constants  $f, g$  provide one extra parameter for a single elliptic curve with a redundancy under scaling,

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<sup>5</sup>Technically, manifolds with  $K$  in a torsion class,  $nK = 0$ , are sometimes classified as Calabi-Yau, but we do not worry about such issues here.

the homogeneous functions  $f(w, v), g(w, v)$  have four redundant degrees of freedom through general linear transformations on the homogeneous coordinates  $w, v$ . Thus, the number of complex degrees of freedom for the set of elliptically fibered K3 surfaces (with section) is  $22 - 4 = 18$ , in agreement with the number of degrees of freedom found in the heterotic construction. One additional scalar field is associated with the volume of the base  $B = \mathbb{P}^1$ .

### 3.4.4 Gauge groups and singularities

As discussed above, the generic elliptic K3 is smooth. The resulting F-theory vacuum is an 8D supergravity theory with an abelian gauge group  $U(1)^{20}$  (where as usual two of the  $U(1)$ 's lie in the gravity multiplet.) The derivation of the rank of the gauge group from the F-theory picture is actually somewhat subtle. The abelian gauge group factors arise from global structure of the compactification, unlike nonabelian gauge group factors, which, as we discuss below, arise from the geometry in a simpler local fashion. Naively it might seem that the gauge group should be  $U(1)^{24}$ , with each 7-brane associated with a singularity giving a separate  $U(1)$  factor. Global constraints, however, reduce the rank of the gauge group to 20. This cannot be understood easily from a simple supergravity picture, since the 7-branes cannot all be described perturbatively in the same duality frame. The constraint to rank 20 can be understood from the topology of the global K3, or from the point of view of probe branes on the base, which interact with the 7-branes through open strings and string junctions [52, 53]; more geometrically this corresponds to the structure of rational sections of the fibration (Mordell-Weil group) [45, 54], which encode part of the topology of the global K3 space.

Now let us consider the non-generic situation in which several of the 7-branes on the spherical base  $B$  coincide. This leads to a more complicated singularity in the elliptic fibration and often to a singularity in the total space of the K3. At such a singularity of the K3, one or more two-cycles may shrink to a point. When a two-cycle shrinks to a point, the theory develops an extra massless gauge boson. This gives rise to an enhanced gauge group in the 8D supergravity theory. In the M-theory/IIA picture, the extra gauge boson comes from a membrane/D2-brane wrapped on the vanishing cycle [10]. It was shown by Kodaira that the possible types of singularity in a complex surface that can be resolved by a succession of blow-ups can be systematically classified [55, 56]. For each type of singularity there is a corresponding Dynkin diagram, encoding the intersection form on the shrunk two-cycles. In the F-theory picture, these Dynkin diagrams precisely characterize the nonabelian gauge group arising in that singular limit of the K3 surface. Note the chronic sign difference between the intersection form on the two-cycles and the Cartan matrix; this arises from a difference in conventions between the algebraic-geometric and Lie algebra frameworks.

In the following section we work through an explicit example of an F-theory singularity of type  $A_3$ . In general, the singularity type depends upon the degree of vanishing of  $f, g$  and  $\Delta$  at the singular point. For example, when  $f$  and  $g$  are nonvanishing, and  $\Delta$  vanishes to order  $n$ , there is a singularity of the form  $A_{n-1}$ , associated with nonabelian gauge group  $SU(n)$ . The complete list of Kodaira singularity types is given in Table 1. It is an educational exercise to work through the blow-up procedure for some of the different singularities in the

ord $(f)$	ord $(g)$	ord $(\Delta)$	singularity	nonabelian symmetry
$\geq 0$	$\geq 0$	0	none	none
0	0	$n$	$A_{n-1}$	$SU(n)$
$\geq 1$	1	2	none	none
1	$\geq 2$	3	$A_1$	$SU(2)$
$\geq 2$	2	4	$A_2$	$SU(3)$
2	3	$n+6$	$D_{n+4}$	$SO(8+2n)$
$\geq 2$	$\geq 3$	6	$D_4$	$SO(8)$
$\geq 3$	4	8	$E_6$	$E_6$
3	$\geq 5$	9	$E_7$	$E_7$
$\geq 4$	5	10	$E_8$	$E_8$

Table 1: Table of singularity types for elliptic surfaces and associated nonabelian symmetry groups.

table, and to verify the appearance of the stated Dynkin diagram in the intersection form of the blown-up  $\mathbb{P}^1$ 's. When a (complex) codimension one singularity occurs which has degrees higher than any allowed in the Kodaira table, the geometry cannot be resolved to a space which is locally Calabi-Yau. In Section 4.7, we discuss non-Kodaira singularities of higher codimension, which can lead to physically interesting transitions in the space of theories.

We now have a general picture of how F-theory describes 8D supergravities through compactification on elliptically fibered K3 surfaces. The elliptically fibered K3 surface is described through a Weierstrass equation of the form (87), where  $f, g$  are degree 8 and degree 12 polynomials respectively on the base  $\mathbb{P}^1$  (sections of  $\mathcal{O}(-4K), \mathcal{O}(-6K)$  respectively in the global picture). Generically the gauge group is  $U(1)^{20}$ , though as the moduli vary the singularities associated with vanishing of  $\Delta$  from (89) can coincide, giving more complicated singularity types and enhancing the 8D gauge group to include nonabelian factors.

As a global example of an F-theory vacuum on an elliptically fibered K3 with large symmetry group, we describe the theory with  $E_8 \times E_8$  gauge symmetry, following Morrison and Vafa [45]. Choose

$$f = \alpha z^4, \quad g = z^5 + \beta z^6 + z^7. \quad (93)$$

This gives

$$\Delta = 27z^{10} + \cdots + 27z^{14}. \quad (94)$$

This Weierstrass model has  $E_8$  singularities at  $z = 0, \infty$ , as can be verified from Table 1.

We can now address the question of which gauge groups  $G$  can be realized in eight dimensions through an F-theory compactification on an elliptically fibered K3. The condition that the K3 be elliptically fibered and have a section identifies two cycles  $f, s$  on the total space with intersection products  $f \cdot f = 0, f \cdot s = 1, s \cdot s = -2$ . The linear combinations  $f, s + f$  then have intersection products in K3 given by a copy of  $U$ . The elliptic fibration thus removes a factor of  $U$  from  $H_2(K3; \mathbb{Z}) = \Gamma^{3,19}$ , giving  $\Gamma^{2,18}$  [41]. In this remaining lattice, we can shrink any combination of two-cycles satisfying  $c \cdot c = -2$  to get nonabelian gauge bosons. The set of nonabelian gauge groups that can be realized is then, just as in

the heterotic theory, the set of  $G$  such that

$$G : -\Lambda_G \hookrightarrow \Gamma^{2,18}. \quad (95)$$

Again, as in the heterotic theory, the rank of the gauge group when all  $U(1)$  factors are included is always 20.

### 3.4.5 Example: $A_3$ singularity

It is helpful to study the geometry of a particular case to understand the general principles of the Kodaira classification. Consider an elliptic fibration over a local patch in  $\mathbb{C}$  parameterized by  $w$ , given by the Weierstrass form (87) with

$$\begin{aligned} f &= -\frac{1}{3} - w^2 \\ g &= \frac{2}{27} + \frac{1}{3}w^2. \end{aligned} \quad (96)$$

While  $f$  and  $g$  are both nonvanishing at  $w = 0$ , a small calculation shows that the discriminant vanishes to order  $w^4$

$$\Delta = 4f^3 + 27g^2 = -w^4 - 4w^6. \quad (97)$$

At  $w = 0$ ,  $F = -y^2 + x^3 - x/3 + 2/27$ , which has a singularity at  $x = -1/3, y = 0$ . To simplify the analysis, it is convenient to change coordinates

$$x \rightarrow x + \frac{1}{3} \quad (98)$$

to move the singularity to  $x = 0$ . The Weierstrass equation then becomes

$$F = -y^2 + x^3 + x^2 - w^2x = 0. \quad (99)$$

This gives a local equation for the complex surface described by the elliptic fibration in coordinates  $(x, y, w) \in \mathbb{C}^3$ . The surface has a singularity at  $x = y = w = 0$ . This singularity can be resolved by blowing up the codimension one singularity repeatedly until the space is smooth. We can do this by working in a sequence of coordinate charts containing the various blow-ups. We refer to the local chart where the surface is defined through (99) as Chart 0.

#### Chart 1:

To resolve the singularity in chart 0, we blow up the singular point. We replace the point  $(0, 0, 0)$  with a  $\mathbb{P}^2$  given by the set of limit points described by homogeneous coordinates  $[x : y : w]$  along curves approaching  $(0, 0, 0)$ . The new space produced by the blow-up process can be described as a subspace of  $\mathbb{P}^2 \times \mathbb{C}^3$ , where homogeneous coordinates  $[u : v : t]$  on the  $\mathbb{P}^2$  satisfy the relations

$$uy = vx, \quad uw = tx, \quad vw = ty. \quad (100)$$

When one or more coordinates  $x, y, w$  are nonzero a unique point in the  $\mathbb{P}^2$  is determined,  $[u : v : t] \sim [x : y : w]$ . At the point  $(x, y, w) = (0, 0, 0)$ , however, (100) imposes no conditions on the coordinates  $u, v, t$ .

This blow-up process can be described in different coordinate charts on the  $\mathbb{P}^2$ . We choose, for example, a local chart where  $w \neq 0$ . In this chart we cannot have  $t = 0$  (or  $u = v = 0$  also from (100)), so we fix the homogeneous coordinates on  $\mathbb{P}^2$  using  $t = 1$ . We then have

$$x = uw, \quad y = vw. \quad (101)$$

Relabeling  $u \rightarrow x_1, v \rightarrow y_1$ , this chart (Chart 1) is realized by changing coordinates to

$$(x, y, w) = (x_1 w_1, y_1 w_1, w_1). \quad (102)$$

For  $w_1 \neq 0$ , there is a unique point  $(x, y, w)$  in the original chart (chart 0) corresponding to each point  $(x_1, y_1, w_1)$  in the new chart. For  $w_1 \rightarrow 0$ , however, there is a set of points  $(x_1, y_1, 0)$  in the new chart that all correspond to  $(0, 0, 0)$  in Chart 0. This gives a local patch on the new  $\mathbb{P}^2$  formed from the blow-up, containing the points  $[x_1 : y_1 : 1]$ . In the coordinates of Chart 1, the local equation (99) becomes

$$F = (-y_1^2 + x_1^2 + x_1^3 w_1 - w_1 x_1) w_1^2 = 0. \quad (103)$$

The  $\mathbb{P}^2$  that is added through the blow-up process is known as the *exceptional divisor* associated with the blow-up. The resulting equation (103) is reducible and contains two copies of the exceptional divisor at  $w_1 = 0$ . Factoring out the overall  $w_1^2$  from (103) (*i.e.*, removing the copies of the exceptional divisor), we have the equation for the *proper transform* of the original space

$$F_w = -y_1^2 + x_1^2 + x_1^3 w_1 - w_1 x_1 = 0. \quad (104)$$

This equation describes the surface in chart 1 after the original singularity at  $(x, y, w) = (0, 0, 0)$  has been blown up. (We use the subscript on  $F$  to denote the coordinate chart used for the blow-up.) The intersection of the space defined through (104) with the exceptional divisor at  $w_1 = 0$  gives the exceptional divisor on the surface, which is generally a curve or set of curves associated with blowing up the point at  $w_1 = 0$  within the surface defined by  $F$ . At  $w_1 = 0$ , (104) becomes  $-y_1^2 + x_1^2 = 0$

$$y_1 = \pm x_1. \quad (105)$$

This defines a pair of curves in  $\mathbb{P}^2$ , which we call  $C_1^\pm$ . The equation (104) still contains a singularity at the point  $(x_1, y_1, w_1) = (0, 0, 0)$ , where the curves  $C_1^\pm$  cross. So we must again blow up the singularity to produce a smooth space.

## Chart 2:

We replace the singular point in Chart 1 with another exceptional divisor  $\mathbb{P}^2$ , this time using the local coordinates

$$(x_1, y_1, w_1) = (x_2, y_2 x_2, w_2 x_2) = 0. \quad (106)$$



Figure 6: Exceptional curves resolving an  $A_3$  singularity, depicted as (a) geometry of the exceptional divisors, which take the form of  $\mathbb{P}^1 = S^2$ 's with associated intersections, (b) associated Dynkin diagram.

After removing two copies of the exceptional divisor  $x_2 = 0$  we get the new local equation

$$F_{wx} = -y_2^2 + 1 + x_2^2 w_2 - w_2 = 0. \quad (107)$$

This gives another exceptional curve  $C_2$ , associated with the intersection of (107) with the exceptional divisor  $x_2 = 0$

$$C_2 = \{(x, y, w) : x = 0, w = 1 - y^2\}. \quad (108)$$

Since (107) has no further singularities, we have completely resolved the local singularity and have a smooth space in coordinate chart 2.

From the way in which the exceptional curves  $C_1^\pm, C_2$  intersect, we identify the  $A_3$  form of the singularity found by Kodaira. To compute the intersections, we write (105) in terms of coordinates in chart 2

$$y_2 x_2 = \pm x_2 \Rightarrow y_2 = \pm 1, \quad (109)$$

which combined with  $w_1 = w_2 x_2 = 0$  gives the points  $[1 : \pm 1 : 0]$  in homogeneous coordinates on  $C_2 = \mathbb{P}^2$ , showing that  $C_1^\pm$  each intersect  $C_2$  at a single point but do not intersect one another. The intersections of these curves are shown graphically in Figure 6, along with the associated Dynkin diagram. In principle, we would need to check all coordinate charts at each step to be sure we have not missed any additional singularities. Here we have chosen charts that completely resolve the original singularity.

The example just described is an  $A_3$  singularity. The shrinking two-cycles in the K3 describe an extra set of massless gauge bosons that extend the gauge group to a nonabelian group. In particular, the cycles  $C_1^\pm, C_2$  form the simple roots of the Lie algebra  $SU(4)$ . By taking linear combinations of these cycles we can produce all the nonzero roots of  $SU(4)$ , each of which corresponds to a curve in the K3 with  $c^2 = -2$ . When this type of singularity occurs at a point in a global elliptically fibered K3 compactification of F-theory, the resulting gauge group for the 8D theory has an  $SU(4)$  factor. The rank of the group stays unchanged in such an enhancement, so the total rank is still 20.

### 3.5 The space of 8D supergravities

We have now seen that two very different string-theoretic constructions give rise to the same space of 8D supergravity theories. For both the heterotic and F-theory constructions, there is a low-energy 8-dimensional theory with one supergravity multiplet and vector multiplets forming a rank 20 gauge group. At a generic point in this moduli space the gauge group is  $U(1)^{20}$ , but at special loci in the moduli space the gauge group is enhanced. The possible nonabelian gauge groups that can be realized in each description are precisely those whose root lattices can be embedded (with a factor of  $-1$ ) into  $\Gamma^{2,18}$ . The moduli space of theories has 37 dimensions, parameterized by the scalars in the gravity and vector multiplets. In both the heterotic and F-theory descriptions, this moduli space of theories is a connected space.

The hypothesis that the heterotic string compactified on the torus  $T^2$  and F-theory compactified on K3 give rise not only to the same low-energy supergravity theory, but to the same physical theory at the nonperturbative level, asserts the existence of a duality symmetry between these two classes of string vacua. This duality symmetry, first formulated by Vafa [43], underlies many duality symmetries relating lower-dimensional theories. While, as we have seen, this duality symmetry holds at the level of the field content and symmetries of the theories, a complete mathematical proof of this and all other string duality symmetries is still lacking—in large part because there is as yet no mathematically complete definition of string theory. Nonetheless, the duality between these two ostensibly very different realizations of 8D supergravity theories has been explored in a number of ways beyond what we have described here. An explicit correspondence between the moduli has been identified for some classes of heterotic compactifications [45, 57]. The duality between F-theory on K3 and type I string theory on  $T^2$  was explored in detail by Sen [58]. Protected one-loop corrections to the effective action on the heterotic side have also been matched from the geometry of F-theory [59, 60, 61, 62], suggesting some deeper geometric structure underlying these theories. All evidence suggests that the heterotic and F-theory compactifications are physically equivalent; this suggests in turn that there may be a unique quantum supergravity theory in 8 dimensions with minimal supersymmetry.

Classically, we can couple the 8D supergravity theory to any number of vector multiplets, and the restriction to a rank 20 gauge group that embeds through eq. (95) is not apparent. As for ten-dimensional  $\mathcal{N} = 1$  supergravity, we would like to understand whether quantum consistency restricts the gauge group in a way that matches the range of models realized in string theory. There is at this time no complete argument from the point of view of the macroscopic supergravity theory that restricts the gauge group of the theory in this way. It does seem plausible, however, that quantum consistency of the supergravity theory may impose additional constraints that have not yet been fully elucidated. One possibility is that extra consistency conditions on the gauge group may arise from supersymmetric constraints on the geometric structure of the moduli space.

While space-time anomalies in the 8D theories are not restrictive, we can also consider anomalies on the world-volume of solitonic string excitations of the theory; such anomalies may impose additional constraints on the theory. Every 8D supergravity theory has an

antisymmetric two-form field  $B_{\mu\nu}$  in the spectrum. There are classical solutions describing stringlike black brane excitations of the theory that couple to the  $B$  field. While there is no proof that quantum excitations of this kind of string soliton must be included in the complete quantum gravity theory for such an 8D supergravity, it seems likely that this is the case. Locally, a classical solution describing a small loop of such string represents a small deformation from the flat space-time background. Some general arguments for the conclusion that quanta of any object carrying an allowed conserved charge must be present in any supergravity theory were recently stated by Banks and Seiberg [63].

A hint for how world-volume anomaly conditions on a solitonic string may place constraints on the set of allowed theories can be seen in 10 dimensions. Anomaly cancellation in the world-volume theory of  $N$  D-strings in the type I description of the  $SO(32)$  supergravity theory was analyzed by Banks, Seiberg and Silverstein [64]. This world-volume theory has  $(0, 8)$  supersymmetry in 2 dimensions. The theory is chiral and carries a  $Spin(N)$  gauge symmetry in its world-volume, which would be inconsistent without cancellation of the gauge anomaly. The theory contains a vector multiplet with 8 left-moving chiral fermions in the adjoint representation of the gauge group. There is a matter multiplet containing 8 bosons corresponding to transverse fluctuations of the string and 8 right-moving fermions in the symmetric tensor representation of the gauge group. There are also 32 left-moving fermions in the singlet representation of the gauge group, which can be associated in the string theory picture with open strings stretching from the D-string to the space-filling type I D9-branes that generate the  $SO(32)$  gauge group (these D9-branes and the type I picture are discussed in more detail in the following section). For the gauge anomaly to cancel we must have

$$\sum x_R A_R = 0 \quad (110)$$

where  $x_R$  denotes the number of chiral left-moving fermions in the representation  $R$  of the gauge group (right-moving fermions entering with the opposite sign), and  $A_R$  is the constant of proportionality between the trace of  $F^2$  in representation  $R$  and the fundamental representation

$$\text{tr}_R F^2 = A_R \text{tr} F^2. \quad (111)$$

Here, as previously,  $\text{tr}_R$  is the trace in representation  $R$ , with the absence of index on the trace on the RHS indicating the fundamental representation. For  $Spin(N)$ , we have  $A_{\text{adj}} = N - 2$  and  $A_{\text{sym}} = N + 2$ . Denoting the number of left-moving fermions in the singlet representation by  $2r = 32$  where  $r$  is the rank of the space-time gauge group, the anomaly cancellation condition is

$$8(N - 2) + 2r - 8(N + 2) = 0, \quad (112)$$

which vanishes precisely when  $r = 16$ . This confirms the consistency of the  $SO(32)$   $\mathcal{N} = 1$  theory in 10 dimensions, and can be interpreted as a constraint on the rank of the space-time gauge group. While this argument is formulated in the language of type I string theory, it should be possible to reproduce the analysis from the point of view of the low-energy theory on the D1-brane itself. By analyzing fluctuations around the solitonic string solution, the rank of the space-time gauge group should thus be fixed to be 16 by cancellation of the

world-volume anomaly. Note that this argument may become more subtle for more general space-time gauge groups, in particular for the  $E_8 \times E_8$  theory.

It has been suggested by Uranga [65] that this kind of argument can be applied in supergravity theories with 16 supercharges in fewer dimensions. If correct, this could lead to a demonstration that the world-volume theory of the solitonic string would be anomalous in any  $D$ -dimensional supergravity theory with 16 supercharges unless the rank of the space-time gauge group is fixed to be  $16 + 2 \times (10 - D) = 36 - 2D$ . In particular, in 8D the gauge group would need to be rank 20. While plausible in schematic form, the details of this argument have not been worked out. In particular, as we have seen, in eight dimensions, the 18 vector multiplets in the Cartan algebra of the gauge group appearing in the theory are on an equal footing, and should all play the same role in the world-volume theory of the solitonic string solution. Thus, we expect a cancellation between the extra vector multiplets and the  $U(1)$  factors in the supergravity multiplet. Such a cancellation is plausible since the graviphoton has different properties from the other vector fields; for example, the couplings of the  $BF^2$  terms for the  $U(1)$  factors in the different multiplets have opposite signs. But a detailed proof that this works out is still lacking, and is left as a challenge for future work. Proving by such an argument that the rank of the gauge group in eight dimensions is constrained is also not enough to prove that the set of consistent theories is precisely those given by string theory. There are groups, such as  $SU(2)^{18} \times U(1)^2$  that are of rank 20 and yet cannot be embedded into  $\Gamma^{2,18}$  as in (72). A proof of the stronger embedding constraint from the point of view of the solitonic strings, combined with the known string constructions of supergravity theories with 16 supercharges through toroidal compactification of the 10D theory, would amount to a proof of string universality for this class of supergravity theories. This could lead to similar conclusions for 4D theories with  $\mathcal{N} = 4$  supersymmetry and 6D theories with  $\mathcal{N} = (1, 1)$  supersymmetry as well as the 8D theory with  $\mathcal{N} = 1$  supersymmetry, and would show that the rank and number of possible nonabelian gauge groups for these theories is finite in each case.

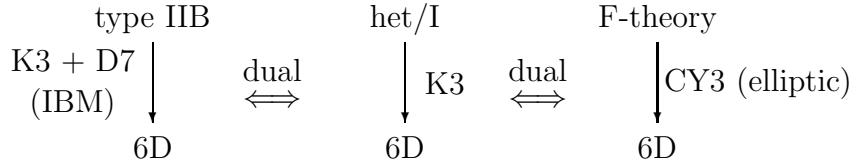
The story just outlined must be incomplete in at least some respects. One complication that must be addressed is the existence of orbifold string compactifications that give rise to theories with 16 supercharges and gauge groups of lower rank. For example, the CHL string [66, 40, 67] gives a theory in eight dimensions with only 10 vector multiplets, and another class of heterotic orbifolds gives rise to an 8D theory with 2 vector multiplets [4]. A complete argument for a bound on the rank of the gauge group in 8D supergravity theories would need to be compatible with the presence of these other discrete structures for the spectrum. Another interesting question is whether these discrete families of string vacua can be smoothly connected in some way to the moduli space of 8D supergravity theories with rank 20 gauge group.

## 4 Supergravity and String Vacua in Six Dimensions

We now turn to supergravity theories in six dimensions. We again focus on theories with the minimum amount of supersymmetry, where the most interesting new phenomena arise. In

six dimensions the SUSY generators are chiral, and there are theories with  $(2, 2)$ ,  $(2, 0)$ ,  $(1, 1)$ , and  $(1, 0)$  supersymmetry. The  $(2, 2)$  theory with maximal supersymmetry arises from compactification of 10D type II supergravity on a torus, and the field content of the theory is uniquely constrained by the supersymmetry structure. The theories with  $(1, 1)$  supersymmetry are in the class of theories with 16 supercharges discussed at the end of the previous section. Theories with  $(2, 0)$  supersymmetry are strongly constrained by anomalies and correspond to the theories realized through compactification of the type II theory on a K3 surface [68]. Theories with  $(1, 0)$  supersymmetry have the richest structure. In particular, these theories can contain matter fields that transform in a variety of representations of the gauge group. These are the 6D theories on which we focus in these lectures. These supersymmetric theories can all be formulated in 6D Minkowski space. While in four dimensions, there are supersymmetric models in AdS space, and gauged supergravity theories with stable supersymmetric backgrounds, such models do not occur in six dimensions [69], so all the supersymmetric 6D theories of interest admit Minkowski vacua.

We begin by describing the constraints from supergravity and then consider string constructions. There is a much wider range of possible string constructions for 6D  $\mathcal{N} = 1$  supergravity theories than for 8D supergravities. We consider several approaches here, adding intersecting brane models on a K3 compactification of type IIB to our repertoire, and explaining the additional complications involved in heterotic and F-theory constructions beyond those encountered in 8D compactifications.



In Section 4.1 we describe the basic structure of 6D supergravity theories with minimal supersymmetry, and in Section 4.2 we characterize the space of theories that satisfy the 6D Green-Schwarz anomaly cancellation conditions. We then introduce some basic aspects of orientifolds in Section 4.3, which we use to describe intersecting brane models in Section 4.4 and magnetized brane models, which are equivalent to compactifications of the heterotic string, in Section 4.5. We briefly describe a variety of additional approaches to heterotic and type I/II constructions that have been used for 6D vacua in Section 4.6. We describe F-theory constructions in six dimensions in Section 4.7. We show in Section 4.8 that the close relationship between a lattice determined by anomaly cancellation in the low-energy theory and the mathematical structure of F-theory allows us to use the data from a low-energy theory to characterize the topological structure of any corresponding F-theory compactification. This is helpful in characterizing the global structure of the set of possible theories. We summarize the current state of knowledge regarding 6D gravity theories with minimal supersymmetry in Section 4.9.

Multiplet	Matter Content
SUGRA	$(g_{\mu\nu}, B_{\mu\nu}^+, \psi_\mu^-)$
Tensor (T)	$(B_{\mu\nu}^-, \phi, \chi^+)$
Vector (V)	$(A_\mu, \lambda^-)$
Hyper (H)	$(4\varphi, \psi^+)$

Table 2: 6D  $(1, 0)$  supersymmetry multiplets

## 4.1 Six-dimensional gravity with $\mathcal{N} = (1, 0)$ supersymmetry

We focus on the massless spectrum of six-dimensional supergravity theories. There are four massless supersymmetry multiplets that appear in  $\mathcal{N} = (1, 0)$  theories with 8 supercharges. These multiplets are summarized in Table 2. The supergravity multiplet contains, in addition to the metric, a bosonic self-dual two-form field  $B_{\mu\nu}^+$ . There are also tensor multiplets that contain anti-self-dual two-form fields  $B_{\mu\nu}^-$  as well as a single scalar field. In general an  $\mathcal{N} = 1$  supergravity theory can have any number  $T$  of tensor multiplets, although the theory only has a Lagrangian description when  $T = 1$ . The two-form fields  $B^\pm$  transform under an  $SO(1, T)$  action that also transforms the scalar fields in the tensor multiplets. These scalar fields parameterize a  $T$ -dimensional moduli space  $SO(1, T)/SO(T)$  that is closely analogous to the moduli space for toroidal compactifications (65). (A further discrete quotient by a duality symmetry group must be taken in the quantum theory, as discussed further below.) As in higher dimensions, the vector multiplet contains the 6D gauge field and a chiral gaugino field. The gauge group of the theory in general takes the form

$$G = G_1 \times G_2 \times \cdots \times G_k \times U(1)^n / \Gamma \quad (113)$$

where  $G_i$  are simple nonabelian gauge group factors, and  $\Gamma$  is a discrete group. The matter hypermultiplets in 6D supergravity theories live in a manifold with a quaternionic Kähler structure. These hypermultiplets can transform in an arbitrary representation (generally reducible) of the gauge group.

To summarize, the discrete data characterizing the field content and symmetries of a 6D  $\mathcal{N} = 1$  supergravity theory consist of the following:

$T$ : the (integer) number of tensor multiplets

$G$ : the gauge group of the theory; we denote by  $V$  the number of vector multiplets in the theory.

$\mathcal{M}$ : the representation of  $G$  characterizing the matter content of the theory. We denote by  $H$  the number of hypermultiplets (including uncharged multiplets) in the theory.

A complete description of the theory would involve further information such as the metric on the scalar moduli space and higher-derivative terms in the action. We do not address this more detailed structure in these lectures. Understanding the extent to which this structure is uniquely determined by supersymmetry and quantum consistency is an interesting direction for future research. The Lagrangian for 6D supergravity theories with one tensor multiplet

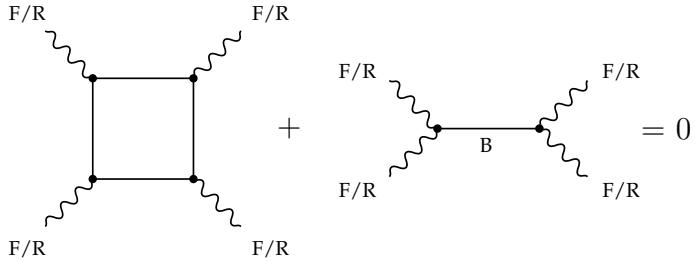


Figure 7: Green-Schwarz anomaly cancellation in six dimensions cancels one-loop anomalies through tree-level diagrams mediated by self-dual and anti-self-dual  $B$  fields.

( $T = 1$ ) was originally described by Nishino and Sezgin [70, 71], and the field equations for models with multiple tensors were developed by Romans [72].

The question we now want to address is: What combinations of  $T, G$ , and  $\mathcal{M}$  are allowed in a consistent 6D supergravity theory? *i.e.*, what is the space  $\mathcal{G}^{6D, \mathcal{N}=1}$ ? To begin to answer this question we consider the known quantum constraints on this class of theories.

## 4.2 Anomalies and constraints on supergravity in 6D

The structure of quantum anomalies in six dimensions is very similar to that in ten dimensions. The chiral fields of the theory that contribute to anomalies are the self-dual and anti-self-dual two-form fields  $B_{\mu\nu}^\pm$ , the gravitino  $\psi_\mu^-$ , gauginos  $\lambda^-$ , and the chiral fermions  $\chi^+$  and  $\psi^+$  from the tensor and hyper multiplets. The anomaly is characterized by an 8-form anomaly polynomial  $I_8(R, F)$ . The anomaly arises from one-loop “box” diagrams with 4 external gauge bosons or gravitons. The Green-Schwarz mechanism again comes into play to cancel anomalies through tree diagrams mediated by an exchange of  $B$  fields [73, 74] (See Figure 7). The story in six dimensions is complicated, however, by the presence of multiple  $B$  fields. The generalization of the Green-Schwarz mechanism including multiple  $B$  fields was worked out by Sagnotti. The 6D gravitational, nonabelian gauge, and mixed gauge-gravitational anomalies cancel when the 8-form  $I_8$  factorizes in the form [75, 76, 77]

$$I_8 = \frac{1}{2} \Omega_{\alpha\beta} X_4^\alpha X_4^\beta \quad (114)$$

where

$$X_4^\alpha = \frac{1}{2} a^\alpha \text{tr} R^2 + \sum_i b_i^\alpha \left( \frac{2}{\lambda_i} \text{tr} F_i^2 \right). \quad (115)$$

Here,  $\Omega_{\alpha\beta}$  is a signature  $(1, T)$  inner product,  $a^\alpha$  and  $b_i^\alpha$  are vectors in  $\mathbb{R}^{1, T}$ , and  $\lambda_i$  are normalization constants for the simple group factors  $G_i$  appearing in  $G$ , where for example  $\lambda_{SU(N)} = 1, \lambda_{E_8} = 60, \dots$ . We have not written the anomaly conditions for  $U(1)$  gauge factors. These take a similar but slightly more complicated form [78, 79, 80]. We do not treat  $U(1)$  factors systematically in these lectures; they add some technical complications to the story but do not play an important role in the main points we wish to emphasize here.

It may be helpful to consider a special case of the anomaly conditions where the factorization takes a particularly simple form. When  $T = 1$ , there is always a basis for  $\mathbb{R}^{1, T}$

where

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad a = (-2, -2), \quad \text{and } b = \frac{1}{2}(\alpha, \tilde{\alpha}). \quad (116)$$

The anomaly factorization condition can then be written as

$$I_8 = X^1 X^2 = (\text{tr}R^2 - \sum_i \alpha_i \text{tr}F_i^2)(\text{tr}R^2 - \sum_i \tilde{\alpha}_i \text{tr}F_i^2) \quad (117)$$

This form of the anomaly cancellation condition appears in much of the literature on 6D models with  $T = 1$ .

The anomaly cancellation conditions are not the only constraints that a 6D supergravity theory must satisfy for consistency. As observed by Sagnotti [75], there is also a constraint coming from the condition that the gauge fields have kinetic terms with the proper (negative) sign. Wrong-sign gauge field kinetic terms would lead to an instability in the theory. As in 10D, the gauge kinetic term is related by supersymmetry to the  $BF^2$  coupling that plays a role in the Green-Schwarz mechanism. The gauge kinetic term is thus proportional to  $-j \cdot b \text{tr}F^2$ . The sign constraint on this gauge kinetic term gives the condition that  $j \cdot b > 0$ . This constraint plays an important role in restricting the set of possible consistent supergravity theories in six dimensions. We discuss other quantum consistency constraints later in this section and in Section 4.9.

Anomaly cancellation through (114) imposes a set of conditions that relate the field content of the theory and the vectors  $a, b_i$  appearing in the anomaly polynomial. Each type of term in the anomaly polynomial ( $R^4, F^2 R^2, \dots$ ) must cancel separately. This gives the following conditions

$$R^4 : \quad H - V = 273 - 29T \quad (118)$$

$$F^4 : \quad 0 = B_{Adj}^i - \sum_R x_R^i B_R^i \quad (119)$$

$$(R^2)^2 : \quad a \cdot a = 9 - T \quad (120)$$

$$F^2 R^2 : \quad a \cdot b_i = \frac{1}{6} \lambda_i \left( A_{Adj}^i - \sum_R x_R^i A_R^i \right) \quad (121)$$

$$(F^2)^2 : \quad b_i \cdot b_i = \frac{1}{3} \lambda_i^2 \left( \sum_R x_R^i C_R^i - C_{Adj}^i \right) \quad (122)$$

$$F_i^2 F_j^2 : \quad b_i \cdot b_j = 2 \sum_{R,S} x_{RS}^{ij} A_R^i A_S^j, \quad i \neq j \quad (123)$$

In these expressions,  $A_R$  is the set of group-theory coefficients for each representation defined by (111), and similarly  $B_R$  and  $C_R$  are defined through

$$\text{tr}_R F^4 = B_R \text{tr}F^4 + C_R (\text{tr}F^2)^2. \quad (124)$$

Rep.	Dimension	$A_R$	$B_R$	$C_R$
$\square$	$N$	1	1	0
Adjoint	$N^2 - 1$	$2N$	$2N$	6
$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\frac{N(N-1)}{2}$	$N-2$	$N-8$	3
$\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$	$\frac{N(N+1)}{2}$	$N+2$	$N+8$	3
$\begin{smallmatrix} \square \\ \square \\ \square \\ \square \end{smallmatrix}$	$\frac{N(N-1)(N-2)}{6}$	$\frac{N^2-5N+6}{2}$	$\frac{N^2-17N+54}{2}$	$3N-12$

Table 3: Values of the group-theoretic coefficients  $A_R, B_R, C_R$  and dimension for some representations of  $SU(N)$ ,  $N \geq 4$ . For  $SU(2)$  and  $SU(3)$ ,  $A_R$  is given in table, while  $B_R = 0$  and  $C_R$  is computed by adding formulae for  $C_R + B_R/2$  from table with  $N = 2, 3$ .

The numbers  $x_R^i$  represent the number of matter fields in the  $R$  representation of  $G_i$ , and similarly  $x_{RS}^{ij}$  is the number of matter fields transforming in the  $R \times S$  representation of  $G_i \times G_j$ . Values of the group theory coefficients  $A_R, B_R, C_R$  are straightforward to compute using elementary group theory methods; a table of values for a few representations of  $SU(N)$  are given in Table 3, and more extensive lists and methods for deriving these factors can be found in various relevant papers [78, 81, 82].

The relations (120-123) determine the inner products of the vectors  $a, b_i$  in terms of the matter content of the theory. It can be proven using group theory identities [77] that these anomaly cancellation conditions<sup>6</sup> automatically lead to integral values for the inner products  $a \cdot a, a \cdot b_i, b_i \cdot b_j$ . Thus, the vectors  $a, b_i$  define an integral lattice

$$\Lambda \subset \mathbb{R}^{1,T}. \quad (125)$$

We refer to  $\Lambda$  as the *anomaly lattice* of the 6D theory. (Note that the basis chosen in (116) for  $T = 1$  is not always the basis in which the basis vectors for the lattice are integral.)

**Example:**

Consider a theory with one tensor multiplet ( $T = 1$ ), nonabelian gauge group

$$G = SU(N) \quad (126)$$

and charged matter content

$$\text{matter} = 2 \times \begin{smallmatrix} \square \\ \square \end{smallmatrix} (\mathbf{N}(\mathbf{N}-1)/2) + 16 \times \mathbf{N} \quad (127)$$

For each representation  $\mathbf{R}$  of  $G$  listed, the matter content contains one complex scalar field in representation  $\mathbf{R}$  and a corresponding field in the conjugate representation  $\bar{\mathbf{R}}$ . Together these fields form the quaternionic structure needed for the scalar moduli space. (Note that for special representations like the  $\mathbf{2}$  of  $SU(2)$ , the representation is itself quaternionic, so that the conjugate need not be included. In cases like this the field is often referred to

<sup>6</sup>for gauge groups such as  $SU(2)$  and  $SU(3)$  with no fourth order invariant, cancellation of global anomalies [83, 84] is also needed [77]

as a “half hypermultiplet”). The total number of charged hypermultiplets in the theory is  $H = N^2 + 15N$  and the number of vector multiplets is  $V = N^2 - 1$ , so from (118) we have

$$H - V = H_{\text{neutral}} + H_{\text{charged}} - V = H_{\text{neutral}} + 15N + 1 = 244. \quad (128)$$

It follows that  $N \leq 16$ . From Table 3, it is easy to verify that (119) is also satisfied for any  $N$ , and the matrix of inner products defining the anomaly lattice for the theory is given by

$$\Lambda = \begin{pmatrix} a \cdot a & -a \cdot b \\ -a \cdot b & b \cdot b \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 2 & 0 \end{pmatrix}. \quad (129)$$

**Example:**

Now consider a theory with one tensor multiplet ( $T = 1$ ), gauge group

$$G = SU(5) \times SU(6) \quad (130)$$

and matter content

$$\text{matter} = 2 \times (\mathbf{10} (\square), \mathbf{1}) + 1 \times (\mathbf{5}, \bar{\mathbf{6}}) + 10 \times (\mathbf{5}, \mathbf{1}) + 7 \times (\mathbf{1}, \mathbf{6}) + 161 \times (\mathbf{1}, \mathbf{1}). \quad (131)$$

The total number of hypermultiplets is  $H = 303$  and the number of vector multiplets is  $V = 59$ , so  $H - V = 244 = 273 - 29T$ , and (118) is satisfied. From Table 3, it is easy to verify that (119) is also satisfied, and the matrix of inner products defining the anomaly lattice for the theory is given by

$$\Lambda = \begin{pmatrix} a \cdot a & -a \cdot b_1 & -a \cdot b_2 \\ -a \cdot b_1 & b_1 \cdot b_1 & b_1 \cdot b_2 \\ -a \cdot b_2 & b_1 \cdot b_2 & b_2 \cdot b_2 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 2 \\ 0 & -2 & 1 \\ 2 & 1 & 0 \end{pmatrix}. \quad (132)$$

The lattice  $\Lambda \subset \mathbb{R}^{1,T}$  is two-dimensional; we see from (132) that the matrix of inner products is degenerate, and there is a linear relation between the vectors  $-a = 2b_1 + 4b_2$ . In the basis (116), we have  $a = (-2, -2)$ ,  $b_1 = (1, -1)$  and  $b_2 = (0, 1)$ , reproducing the desired inner products.

The anomaly cancellation conditions for 6D supergravity theories place strong constraints on the range of possible consistent theories. It has been proven that when  $T < 9$ , the set of possible gauge groups  $G$  and matter representations  $\mathcal{M}$  charged under the nonabelian gauge group factors is finite [81, 77]. This bound is valid for theories with nonabelian and abelian gauge group factors, although for theories with abelian factors the charges of the matter fields under the  $U(1)$  factors are not constrained to a finite range of possible values by anomaly cancellation [80]. The proof of the finiteness result is slightly technical, but relies at its core on the relation (118), which bounds the number of matter multiplets for a given number of tensor fields and gauge content. Roughly, if the gauge group factors are all of limited dimension ( $\dim(G_i) < D \forall i$ ), then it can be shown that almost all pairs of gauge group factors share hypermultiplets with charge under both factors. This leads to a situation

where  $H \sim \mathcal{O}(k^2)$  and  $V \sim \mathcal{O}(k)$ , where  $k$  is the number of distinct nonabelian factors in the gauge group. As  $k \rightarrow \infty$ , the constraint (118) must be violated. For the details of this argument the reader is referred to the original papers [81, 77]. Anomaly cancellation alone does not rule out the possibility of a finite number of gauge group factors with unbounded dimension. It was shown by Schwarz [85] that there are infinite families of theories with  $T = 1$  that satisfy the anomaly factorization conditions. For example, the theories with gauge group  $G = SU(N) \times SU(N)$  and matter content  $2 \times (\mathbf{N}, \bar{\mathbf{N}})$  satisfy the anomaly cancellation conditions and have  $H - V = 2$  for any  $N$ . There are 5 infinite families of models that satisfy the factorization conditions. For each such family, however, it is possible to prove that everywhere in the moduli space at least one gauge group factor must have the wrong sign on the kinetic term [77]. For example, for the theories just mentioned that were found by Schwarz, it is easy to check that the anomaly conditions give

$$a \cdot b_1 = a \cdot b_2 = 0, \quad b_1^2 = b_2^2 = -2, \quad b_1 \cdot b_2 = 2. \quad (133)$$

From this it follows that, when  $a^2 > 0$ ,

$$a \cdot (b_1 + b_2) = 0 \quad \& \quad (b_1 + b_2)^2 = 0 \quad \Rightarrow \quad b_1 + b_2 = 0 \quad \Rightarrow \quad j \cdot b_1 = -j \cdot b_2. \quad (134)$$

Thus, at least one of the gauge group factors has the wrong sign on the kinetic term for any vector  $j$ . This rules out families of theories with gauge group factors of unbounded dimension, as long as  $T < 9$ . When  $T \geq 9$ , however, the norm of the vector  $a$  is no longer positive definite, (recall  $a^2 = 9 - T$ ), so the above argument no longer works. In fact, there are infinite families of models with  $T = 9$  and greater that satisfy the anomaly cancellation equations and have proper-sign kinetic terms [77].

This gives a basic outline of the set of 6D  $\mathcal{N} = 1$  supergravity theories that satisfy the anomaly cancellation and gauge kinetic term sign constraints. It was recently shown that a further constraint can be placed on the set of 6D supergravity theories that can give consistent quantum theories. The set of allowed charges for objects that couple to the  $B$  fields of the theory form a lattice  $\Gamma$  of signature  $(1, T)$ . This lattice must be integral, from the Dirac quantization condition [86]. The consistency of the dimensional reduction of the theory to 2D or 4D requires that the lattice is furthermore self-dual [87]. This conclusion is compatible with the general mathematical framework for treating (anti)self-dual  $p$ -form fields that has recently been under development [88, 89, 90, 91]. The charge lattice is invariant under a discrete duality group  $G^{1,T} \subset SO(1, T)$ . This reduces the part of the moduli space of the theory parameterized by the scalars  $\phi$  in the tensor multiplets to

$$G^{1,T} \backslash SO(1, T) / SO(T). \quad (135)$$

The vectors  $b_i$  in the anomaly lattice are associated with gauge dyonic strings [92] associated with instantons in each gauge group factor, and thus represent vectors in the charge lattice  $\Gamma$ . There must therefore be an embedding of the lattice spanned by the vectors  $b_i$  into the lattice  $\Gamma$ . The vector  $a$  should also give a vector in the charge lattice  $\Gamma$ , though this is not rigorously proven. The condition that an embedding of the anomaly lattice  $\Lambda$  into  $\Gamma$  is

possible imposes further constraints on the set of allowed 6D theories. For example, consider the theory with  $T = 2$ , gauge group

$$G = SU(N) \times SU(N), \quad (136)$$

and charged matter content

$$2N \times (\mathbf{N}, \mathbf{1}) + 2N \times (\mathbf{1}, \mathbf{N}). \quad (137)$$

The anomaly lattice for this model is spanned by vectors  $-a, b_1, b_2$  with inner products

$$\Lambda = \begin{pmatrix} 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (138)$$

This lattice does not admit an embedding into any unimodular lattice  $\Gamma$ . If it did, the unit cell of  $\Lambda$  would have a volume given by an integer times the volume of the unit cell of  $\Gamma$ . The determinant of the matrix  $\Lambda$  would then be a perfect square. But the determinant is 28, which is not a perfect square, so the lattice does not admit a unimodular embedding and the theory is not a consistent theory. This criterion cuts down further the space of allowed 6D supergravity theories, though it does not eliminate some known infinite families of models that obey all known consistency constraints.

Thus, we have characterized the set of theories  $\mathcal{G}^{6D, \mathcal{N}=1}$  with no currently known inconsistencies. We now turn to string constructions of 6D theories to see what subset of the apparently consistent supergravity theories can be realized in string theory.

### 4.3 Branes and orientifolds

Six dimensions is an excellent playground for systematically developing the methods of string compactification. The range of possibilities for string constructions in 6D is still much simpler than in 4D, but provides for a rich range of possible gauge groups and matter structure. The strong constraints of anomaly cancellation provide a useful mechanism for checking the internal consistency of different approaches to string vacuum constructions, as well as providing a natural framework for relating different constructions.

We describe in some detail three distinct ways of constructing 6D string vacua. We expand upon the development in Section 3 of the heterotic and F-theory approaches, illustrating some of the new features that must be included for compactifications of these types to 6 dimensions. We also introduce the basic structures needed for another class of compactification, the *intersecting brane model* (IBM) construction. We consider intersecting brane models associated with compactifications of type IIB on an orbifold limit of the K3 surface, where D7-branes added to the geometry realize the gauge groups and matter content of the theory. A more detailed introduction to the physics of intersecting brane model constructions, with a focus on the phenomenology of four-dimensional vacua, can be found in the lecture notes by Cvetic and Halverson from this school [93]. We give here a much more

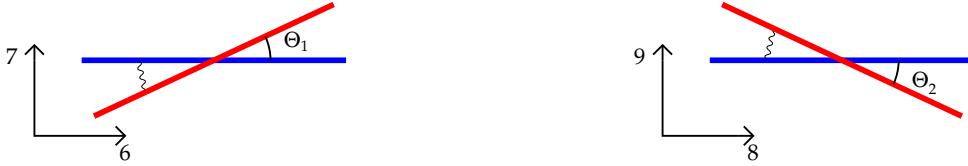


Figure 8: Brane stacks intersecting in two planes preserve some supersymmetry if the intersection angles are equal and opposite in the two planes,  $\theta_1 = -\theta_2$ . Strings stretching between the branes live in the bifundamental representation  $(N, \bar{M}) + (\bar{N}, M)$  of the gauge groups on the two brane stacks.

rudimentary introduction to the subject of IBM's, focusing on how the models generated through this construction fit into the broader set of 6D string vacua.

To understand intersecting brane models and heterotic/type I compactifications in 6D, we must first review some basic aspects of D-branes and orientifolds. These objects are explained clearly and pedagogically in Polchinski's text [7, 8], but we summarize the basic structures involved in order for these lectures to be somewhat self-contained.

The essential feature of a D-brane is that it is a locus in space-time where open strings end. A  $Dp$ -brane in the flat ten-dimensional space-time of type IIA or IIB string theory is a fluctuating hypersurface of spatial dimension  $p$ , whose motion is described by the quantized open strings ending on the brane. When  $N$   $Dp$ -branes are coincident, the D-branes carry a world-volume gauge group  $U(N)$ ; the dynamics of this gauge group and the scalars describing transverse fluctuations of the brane are captured by the physics of the open strings. One approach to describing theories in fewer than 10 dimensions with chiral matter is to use systems of intersecting branes. When two different branes intersect, the strings stretching between the branes give rise to chiral matter fields in the dimensionally reduced theory.

Consider a pair of flat D-branes in  $\mathbb{R}^{1,9}$  that are parallel and coincident in all but 4 spatial dimensions. For example, two D7-branes that are both extended in dimensions 0-5, but each extended in a different two-plane in the four-dimensional space 6789. We can view these branes as intersecting in two different coordinate planes; for example the branes may each extend on different lines in the 67 plane and the 89 plane (See Figure 8). The branes preserve a common supersymmetry if the angles between the branes in the two planes are equal and opposite [94]

$$\theta_1 + \theta_2 = 0. \quad (139)$$

If there are stacks of  $N, M$  coincident branes intersecting, then the open strings connecting the branes produce fields living on the brane intersection locus that transform in the bifundamental representation  $(N, \bar{M}) + (\bar{N}, M)$ . If we compactify the theory down to six dimensions, this will give matter in the low-energy 6D theory transforming in the same representation.

The obstacle to considering compactifications with D-branes is that the D-branes act as sources for the Ramond-Ramond fields. A D-brane that extends in all noncompact di-

mensions of space-time will produce a flux in the transverse compact directions. D-branes that preserve the same supersymmetry will generally carry the same kind of charge. With net D-brane charge, the flux lines will have nowhere to end on the compact space, and the geometry will not admit a solution of the equations of motion. The situation is like a set of positive charges in a compact space; the electric field lines point away from the positive charges, and without negative charges to collect the field lines, a compact solution is impossible. Thus, we need something to cancel the R-R charges of the D-branes in order to use the branes to generate the low-energy gauge group and matter content. One solution to this problem is the *orientifold*. Orientifolds are similar to D-branes, but carry negative tension and negative charge. When orientifolds are included, compactifications with D-branes can preserve SUSY and give rise to interesting low-energy theories<sup>7</sup>.

To understand orientifolds, it is easiest to begin with the space-filling orientifold that relates the type IIB theory to the type I string theory. Consider the transformation  $\Omega$  that acts by a reflection on the internal coordinate of a string

$$\Omega : \sigma \rightarrow -\sigma. \quad (140)$$

This transformation exchanges the right- and left-moving string degrees of freedom  $\Omega : x_L \leftrightarrow x_R$ ,  $\Omega : \alpha \leftrightarrow \tilde{\alpha}$ . This is a symmetry of the type IIB string theory. If we consider only string states that are invariant under this symmetry (essentially gauging the symmetry  $\Omega$  by taking the orbifold of the theory by  $\Omega$ ), then the string spectrum will be simplified. For example, the  $B$  field of the type IIB theory is associated with the first-quantized string states

$$[\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu - \tilde{\alpha}_{-1}^\mu \alpha_{-1}^\nu] |0\rangle. \quad (141)$$

This set of states is projected out by the orientifold action. The theory realized by the orientifold of the type IIB theory is the type I theory, which has no  $B$  field, only the R-R field  $\tilde{B}$ . The type I theory has only one supersymmetry in 10 dimensions. As we know, an  $\mathcal{N} = 1$  theory in 10D must have a gauge group of dimension 496 to be anomaly-free. Thus, some additional structure must provide a gauge group for the type I theory. The projection under  $\Omega$  produces an unoriented string theory. String diagrams with unoriented topology, such as the Möbius strip and Klein bottle, must be included. From the analysis of such diagrams, the type I orientifold background has been shown to be inconsistent without the addition of 16 space-filling D9-branes, whose world-volume theory (after including the orientifold projection) is precisely the gauge group  $SO(32)$  needed for an anomaly-free theory [8]. This can be understood in terms of the presence of a space filling “orientifold 9-plane,” with negative tension and D9-brane charge -16, which cancels the charge of the 16 D9-branes.

By combining the world-sheet symmetry  $\Omega$  with a space-time reflection  $\rho$ , orientifold planes of lower dimension can be produced. One-loop string diagrams show that such an

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<sup>7</sup>Note that the 7-branes on  $S^2$  in the F-theory description of 8D vacua described in Section 3.4.1 evade this problem in a different way; the condition that the product of monodromies cancels on the F-theory base is equivalent to the condition that the charges cancel, which can be done in a supersymmetric way when the total space of the elliptic fibration defining the F-theory model is a Calabi-Yau. Like everything else in string theory, these apparently different constructions are related through duality [58].

orientifold plane of codimension  $d$  carries D-brane charge  $16/2^d$ . One easy way to understand this result is by analyzing toroidal compactification and T-duality for the type I theory. Recall that for a string compactified on a circle  $S^1$  of radius  $R$ , the spectrum of momentum and winding modes is given by

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'} . \quad (142)$$

As discussed in Section 3.3.3, T-duality is the symmetry that exchanges

$$\text{T-duality} : n \leftrightarrow m . \quad (143)$$

T-duality also exchanges  $x_R \leftrightarrow -x_R$ . It follows that T-duality exchanges a D-brane wrapped in the compact direction with one that is unwrapped ( $D_p \leftrightarrow D_{p\mp 1}$ ); the gauge field component  $A_\mu$  on the  $Dp$ -brane wrapped in the compact direction  $\mu$  becomes the transverse scalar field  $X^\mu$  of the unwrapped brane

$$T : A_\mu \leftrightarrow X^\mu . \quad (144)$$

The relation (144) can be understood from the point of view of the string world-sheet [7], or from the world-volume theory of a brane on a compact space [95]. Just as T-duality can reduce or increase the dimension of a D-brane, in a similar fashion T-duality reduces the dimension of a space-filling orientifold plane. The action of  $\Omega : x \rightarrow x$  on  $x = x_R + x_L$  before T-duality becomes the reflection  $\Omega : x \rightarrow -x$  on  $x = -x_R + x_L$  after T-duality. On the dual circle, this action has two fixed points, at  $x = 0, x = \pi R$ . Thus, under T-duality of the type I theory on  $d$  circles in the torus  $T^d$  the 16 D-branes are distributed across  $2^d$  orientifold  $(9-d)$ -planes. This confirms the statement above that each orientifold  $(9-d)$ -plane carries D-brane charge  $-16/2^d$ .

This brief summary of the physics of D-branes and orientifold planes gives us enough background to construct intersecting brane models on compactifications of the type IIB theory to six dimensions.

## 4.4 Intersecting brane models in 6D

Intersecting brane models (IBM's) provide a rich range of examples of string compactifications with a variety of gauge groups and chiral matter. Semi-realistic IBM's in four dimensions have been the subject of much study [96, 93]. Here we will just introduce the basics of the construction in six dimensions to show how some 6D supergravity theories can be realized from this approach. A more detailed analysis of 6D intersecting brane models was given by Blumenhagen, Braun, Körs, and Lüst [48]; the presentation here follows the analysis and notation of that work. Additional features of these models have been developed in work with Nagaoka [97].

While in principle, intersecting brane models on smooth Calabi-Yau manifolds give rise to a very general class of string compactifications, the analysis is greatly simplified by working on a toroidal orbifold. We will use here the orbifold limit of K3 described in Section 3.4.2. Thus, we consider the orbifold of  $T^4$  by the symmetry

$$\rho : z_i \rightarrow -z_i . \quad (145)$$

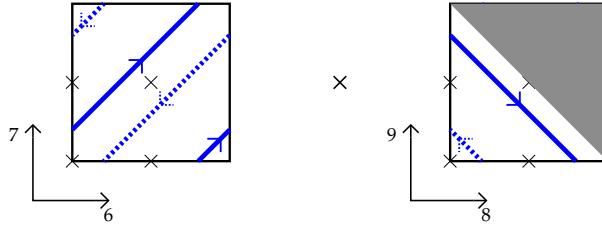


Figure 9: A diagonal D7-brane wrapped on the cycle  $(1, 1)$  on the first  $T^2$  and the cycle  $(1, -1)$  on the second  $T^2$ . The dotted part of the brane is associated with the piece that lifts under the orbifold action  $\rho : z_i \rightarrow -z_i$  to the shaded part of the covering space outside the fundamental domain.

Here, as before,  $z_1, z_2$  are complex coordinates on two  $T^2$  factors of  $T^4$ . We assume that both  $T^2$  factors are rectangular, and have moduli  $\tau_j = iR_j$ ,  $j = 1, 2$ .

In order to include D-branes we must also include orientifolds. We include an orientifold 7-plane (O7-plane) defined through the orientifold action  $\Omega\sigma$ , where  $\sigma$  gives the reflection

$$\sigma : z_i \rightarrow \bar{z}_i . \quad (146)$$

This gives an O7-plane in the 68 directions. Since we are taking the orbifold separately by  $\rho$  and  $\Omega\sigma$ , we must also include the combined orientifold action  $\Omega\rho\sigma$ ; this gives another O7-plane in the 79 directions. As discussed above, there are really 4 copies of each O7-plane in the covering space  $T^4$ , at antipodal points on the perpendicular tori. After taking the orbifold action, the combined set of cycles where the orientifold is wrapped becomes

$$\pi_{O7} = 2(\pi_{68} - \pi_{79}) . \quad (147)$$

Now consider D7-branes that are wrapped on the  $T^4$  as a product of one-cycles on the two  $T^2$  factors with winding numbers

$$(n_1, m_1; n_2, m_2) . \quad (148)$$

For example, a D7-brane wrapped on the cycle  $(1, 1; 1, -1)$  is depicted in Figure 9. The homology class of a brane with winding numbers (148) is

$$\pi = n_1 n_2 \pi_{68} + n_1 m_2 \pi_{69} + m_1 n_2 \pi_{78} + m_1 m_2 \pi_{79} . \quad (149)$$

Associated with each such brane there is an orientifold image with winding numbers (reflected across the horizontal axes)  $(n_1, -m_1; n_2, -m_2)$  and associated homology class  $\pi'$ . We consider a general D7-brane configuration to be composed of stacks of  $N_a$  coincident branes with winding numbers  $(n_1^a, m_1^a; n_2^a, m_2^a)$ . The total D-brane charge for all branes extended in space-time and wrapped around any particular cycle of the compact space must vanish as discussed above. Each O7-plane carries -4 units of D7-brane charge. Combining the branes and their orientifold images, the condition for the charges to vanish on the compact space is then

$$\sum_a N_a n_1^a n_2^a = 8, \quad \sum_a (-N_a m_1^a m_2^a) = 8 . \quad (150)$$

These equations are often referred to as “tadpole cancellation” conditions since they are needed for cancellation of tadpoles in the R-R fields of the theory; similar conditions arise in most brane model constructions. For 6D models, these conditions can be related to the anomaly cancellation conditions of the associated supergravity theory.

If we assume that the branes all preserve a common supersymmetry, then the SUSY condition (139) implies that for all branes in the system

$$\frac{m_1}{n_1} = -\alpha \frac{m_2}{n_2} \quad (151)$$

for a common value of  $\alpha$  that parameterizes the moduli of the two  $T^2$  factors.

A stack of  $N$  diagonal branes that do not coincide with their orbifold counterpart (*i.e.*, branes that are not parallel to the orientifold plane and that do not pass through the orbifold points) carry a  $U(N)$  gauge group. Each such brane is wrapped on a cycle that is topologically a torus and will have a single matter field in the adjoint representation, which describes transverse motions of the brane. As discussed in the previous section, open strings between intersecting branes produce bifundamental fields transforming under the gauge groups on each brane. Symmetric and antisymmetric representations of the group  $SU(N)$  for a particular brane stack will also be produced by intersections of the brane with its orientifold image. A table of the multiplicity of matter representations associated with a pair of branes  $\pi_a, \pi_b$  follows; in each case the representation given,  $R$ , signifies a hypermultiplet with matter in the  $R + \bar{R}$  representation.

representation	multiple
Adj	1
$(\square_a, \bar{\square}_b)$	$\pi_a \cdot \pi_b$
$(\square_a, \square_b)$	$\pi_a \cdot \pi'_b$
	$\frac{1}{2}(\pi_a \cdot \pi'_a + \pi_a \cdot o_7)$
	$\frac{1}{2}(\pi_a \cdot \pi'_a - \pi_a \cdot o_7)$

This completes the specification of the simplest intersecting brane models on K3. Other types of branes can be included, such as branes parallel to the orientifold planes, fractional branes, and branes intersecting the orbifold points [48, 97]; these brane types can include other gauge group factors such as  $Sp(N)$  and introduce additional interesting features that we do not explore here. Another complication that we do not treat in detail here is associated with the  $U(1)$  factors in the gauge group. In general some of the  $U(1)$  factors in an intersecting brane model will be anomalous, and will acquire a mass through the Stückleberg mechanism. As above, we primarily focus here on the nonabelian part of the gauge group, and do not treat the  $U(1)$  factors carefully.

To summarize, in the basic class of 6D IBM models just described, we have a set of stacks of  $N_a$  branes, with each stack carrying a gauge group factor  $U(N_a)$ . The winding numbers for each stack satisfy (151), and the tadpole constraint associated with the total charge of the branes and orientifold plane (on each homology class) is satisfied through the conditions

(150). The set of models that satisfy these conditions is fairly limited. We mention here only the simplest example.

**Example: 6D IBM on K3**

The simplest example is to take a stack of 8 branes with winding numbers  $(1, 1; 1, -1)$  as depicted in Figure 9. This model clearly satisfies the SUSY and tadpole cancellation conditions. The supergravity theory associated with this model has gauge group

$$G = U(8). \quad (152)$$

From (82) it follows that  $\pi \cdot \pi' = \pi \cdot o_7 = 8$ , so the matter content of the theory is

$$\text{matter} = 1 \times (\mathbf{63}) + 8 \times (\mathbf{28}). \quad (153)$$

Considering only charged hypermultiplets, we have  $H_{\text{charged}} - V = 224$ . This leaves 20 uncharged hypermultiplets to saturate the gravitational anomaly constraint, which is precisely the number of scalar hypermultiplets that come from the closed string sector. The  $F^4$  anomaly equation (122) for an  $SU(N)$  gauge group factor under which there are  $f$  fundamental matter representations,  $D$  adjoint representations, and  $A$  antisymmetric representations is

$$f = 2N - 2ND - A(N - 8), \quad (154)$$

which is satisfied for this matter content. In all 6D intersecting brane models of this type, there is a single tensor multiplet in the 6D theory ( $T = 1$ ); the anti-self-dual two-form from the tensor and the self-dual two-form from the gravity multiplet combine to form the two-form field that descends from the 10D  $B_{\mu\nu}$ . Connecting to the general framework for 6D supergravities, we can compute the anomaly lattice of this theory, which is spanned by vectors  $-a, b$  with inner product matrix

$$\Lambda = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \quad (155)$$

This lattice is degenerate, and represents a 1D lattice spanned by the generator  $b = -a$ . We show in Section 4.8 how this data can be used to construct the equivalent F-theory model.

## 4.5 Magnetized brane models and heterotic bundles in 6D vacua

We now consider 6D theories associated with compactifying  $\mathcal{N} = 1$  supergravity on a K3 surface. The general compactification of this type is described by a nonabelian instanton configuration on K3. We primarily consider a simple subclass of these models, where the instantons are characterized by  $U(1)$  fluxes on cycles in the K3 surface. The models in this class were the first 6D compactifications of string theory [73]. We can think of such compactifications from the string point of view either as heterotic or type I string compactifications. We begin from the supergravity point of view and then connect to the type I picture, which in spirit is like a T-dual of the intersecting brane models just considered. As for the intersecting brane models on the orbifold limit of K3 discussed above, all models constructed by compactification on a smooth K3 have  $T = 1$ , with the anti-self-dual and self-dual two-forms combining to form the dimensionally reduced  $B$  field from 10D.

### 4.5.1 D-brane charges on branes and orientifolds

We begin from the point of view of supergravity. In compactifying an  $\mathcal{N} = 1$  10D theory to eight dimensions on a torus, the only additional structure available was the possibility of nontrivial Wilson lines around the cycles of the torus. Compactifying on a non-toroidal Calabi-Yau requires the further introduction of a gauge bundle carrying instantons. To see why this is the case, recall from (25) that the 3-form  $H$  contains Chern-Simons contributions from the Yang-Mills and space-time curvatures. The Bianchi identity in a general background is then generalized from  $dH = 0$  to

$$dH = \text{tr}R \wedge R - \text{Tr}F \wedge F. \quad (156)$$

Since the left-hand side is an exact form, the right-hand side must vanish in cohomology, implying a nonzero instanton number for a compactification on K3

$$\int_S c_2(F) = \frac{1}{8\pi^2} \int_S \text{Tr}F \wedge F = \frac{1}{16\pi^2} \int_S \text{Tr}R \wedge R = -\frac{1}{2} \int_S p_1(R) = 24, \quad (157)$$

where  $c_2$  and  $p_1$  are the second Chern class and first Pontryagin class, respectively<sup>8</sup>. For compactification of an  $\mathcal{N} = 1$  10D theory on K3, then, we must have a background gauge field configuration with instanton number 24.

An alternative perspective on the conclusion that the gauge bundle on K3 must have 24 instantons comes from the type I picture, which can be used when the 10D gauge group is  $SO(32)$  (really  $\text{Spin}(32)/\mathbb{Z}_2$ ). Recall that in the type I picture the gauge group comes from 16 D9-branes superimposed on an orientifold O9-plane. An important aspect of D-brane physics is the presence of Chern-Simons couplings in the world-volume action between powers of the field strength  $F$  and the Ramond-Ramond  $p$ -form potentials [98]. For D9-branes, there is a coupling to the 6-form potential of the IIB and I theories proportional to

$$\int F \wedge F \wedge \hat{C}_6. \quad (158)$$

( $\hat{C}_6$  is the dual to the R-R two-form  $\tilde{B}$ .) This means that an instanton on a D9-brane carries the charge of a D5-brane [99]. This statement, as well as the more general statement that a system of D $p$ -branes can carry charge associated with a D $(p \pm 2k)$ -brane can be understood easily from T-duality on the D-brane world-volume [100]. By taking the world-volume derivative of (144) we have

$$T : \partial A_\mu \leftrightarrow \partial X^\mu. \quad (159)$$

Thus, T-duality can relate, for example, flux  $F = \partial A$  on a D9-brane to the slope  $\partial X$  of a tilted D8-brane in a T-dual picture. The T-dual of a diagonal D8-brane on a torus has both

---

<sup>8</sup>Note the factor of 2 in moving between the vector and adjoint representations of  $SO(1,3)$ , denoted respectively by  $\text{tr}$  and  $\text{Tr}$ . Note also that we have changed normalization relative to (25) so that the instanton number is an integer when expressed in conventional form.

D9-brane and D7-brane charge; the D7-brane charge can be associated with the flux encoded in  $F$  on the D7-brane. Generalizing this picture to multiple T-dualities shows that  $F \wedge F$  on a D $p$ -brane carries D( $p - 4$ )-brane brane charge, etc. (A further generalization to multiple D $p$ -branes, replacing  $\partial \rightarrow \partial + iA_\mu$ , leads to the T-dual statement that  $[X, X]^k$  on a stack of multiple D $p$ -branes carries charge associated with higher-dimensional D( $p + 2k$ )-branes [101, 102, 103].) Through the geometric picture of T-duality as relating fluxes to tilting on branes, we see that intersecting diagonal D7-branes on a torus  $T^4$  can be T-dual to a system of D9-branes carrying D7-brane and D5-brane charges through world-volume fluxes. This relates the intersecting brane models discussed in the previous section to the type I models discussed here. There are some subtleties, however, in this correspondence, since we are really working on K3 and not a torus. Also, both the IBM and magnetized brane models are described in the supergravity approximation where the volume of the compactification space is large, and T-duality maps a large compactification torus to a small dual torus, where the supergravity approximation is no longer valid.

In the type I picture, the need for a gauge field configuration with instanton number 24 follows from structure of the D-brane world-volume theory related to the Chern-Simons couplings (158). From supersymmetry, there are similar terms of the form  $R \wedge R \wedge \hat{C}_6$  in the D9-brane world-volume action [104], and analogous couplings occur for the orientifold plane [105, 106]. The quantization of the coupling is such that each D9-brane wrapped on a K3 carries -1 units of D5-brane charge, while the O9-plane wrapped on K3 carries -8 units of D5-brane charge. Combining one O9-plane and 16 D9-branes gives a total deficit of 24 units of D5-brane charge, or instanton number 24, needed to cancel the  $\hat{C}_6$  tadpole. The term “magnetized brane” refers to  $U(1)$  fluxes  $F$  on the D9-brane world-volume that can be used to realize the needed instanton.

#### 4.5.2 Abelian instantons on K3

To understand a general 6D compactification of the 10D  $\mathcal{N} = 1$  theory, then, we must consider the moduli space of instantons for a nonabelian gauge theory on K3. This leads into a rich and interesting mathematical story, which leads beyond the scope of these lectures. Here we will focus on the simple class of “magnetized brane” models where the instanton structure is completely encoded in fluxes within a commuting set of  $U(1)$ ’s in the SO(32) gauge group. This is the class of 6D models originally studied by Green, Schwarz, and West [73]. By describing the structure of these theories in terms of the homology lattice of K3 with associated intersection product, following work with Kumar [49], we make contact with the general theme of lattice embeddings that we have already encountered in the 8D context, and put these models into a framework that connects with the other classes of 6D vacua.

We begin the discussion with a review of some basic facts about K3. Like any complex surface, K3 has a choice of complex structure

$$\Omega \in H^{2,0}(S) \subset H^2(S) \tag{160}$$

that is fixed up to a scale factor. The complex structure satisfies

$$\int \Omega \wedge \Omega = 0, \quad \int \Omega \wedge \bar{\Omega} \propto \text{Vol}(S) > 0. \quad (161)$$

Writing  $\Omega = x + iy$ , it follows that

$$x \cdot y = 0, \quad x \cdot x = y \cdot y > 0, \quad (162)$$

where in this section we will use  $\cdot$  to denote the wedge product in cohomology or the equivalent intersection form in homology, freely moving back and forth using Poincaré duality. There is a Kähler structure associated with a Kähler form  $J \in H^{1,1}(S)$  that satisfies

$$\int J \wedge \Omega = 0 \Rightarrow J \cdot x = J \cdot y = 0 \quad (163)$$

$$\int J \wedge J \propto \text{Vol}(S) > 0 \Rightarrow J \cdot J > 0 \quad (164)$$

It follows that  $(x, y, J)$  define a positive-definite 3-plane in  $H^2(S; \mathbb{R}) = \mathbb{R}^{3,19}$ .

Now, let us consider fluxes of an  $SO(32)$  gauge theory on K3. Fluxes wrapped around nontrivial two-cycles are quantized, and can be normalized to  $F = 2\pi i f$  where

$$f \cdot f \in 2\mathbb{Z}, \quad f \in H^2(K3, \mathbb{Z}) = \Gamma^{3,19}. \quad (165)$$

We thus naturally identify such a flux with an element of the lattice  $\Gamma^{3,19}$ .

The group  $SO(32)$  contains 16 mutually commuting  $U(1)$  factors (associated with  $SO(2)$  transformations on mutually disjoint pairs of the 32 indices). In the type I picture these 16  $U(1)$ 's are the world-volume gauge fields on each of the 16 D9-branes in the theory. A flux  $f$  on one D9-brane is accompanied by a flux  $-f$  on the orientifold image of that D9-brane. (This is essentially the T-dual of the relationship between diagonal branes and their orientifold images in the intersecting brane models described in Section 4.4.) We consider a configuration of the  $SO(32)$  theory where we have “stacks” of  $N_a$  D9-branes with flux  $f_a$ ,  $a = 1, \dots, K$

$$F = \begin{pmatrix} f_1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & f_1 & & & & & & \\ & & & f_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & f_2 & & & \\ & & & & & & \ddots & & \\ 0 & & & & & & & \ddots & \end{pmatrix} \quad (166)$$

The constraint that the total instanton number is 24 becomes

$$\frac{1}{8\pi^2} \int_s \text{Tr} F \wedge F = 24 \Rightarrow \sum_a N_a f_a \cdot f_a = -24. \quad (167)$$

The constraints from supersymmetry (analogous to (151)) give

$$\int f^a \wedge \Omega = 0 \quad \int f^a \wedge \bar{\Omega} = 0 \quad \int f^a \wedge J = 0 \quad (168)$$

$$\Rightarrow f^a \cdot x = f^a \cdot y = f^a \cdot J = 0. \quad (169)$$

The set of abelian instanton configurations on K3 is thus parameterized by the set of fluxes  $\{f_a\}$ . These fluxes generate an even lattice

$$L \subset \Gamma^{3,19}, \quad (170)$$

which characterizes the theory. In fact, the structure of the gauge group and matter content of any 6D theory constructed in this way are dependent only on the integers  $N_a$  and the matrix of inner products  $m_{ab} = f_a \cdot f_b$ . (Note that the matrix  $m$  may be degenerate, in which case the dimensionality of  $L$  is smaller than that of  $m$ .) As long as the lattice  $L$  is negative-definite, there always exists a perpendicular positive-definite 3-plane in  $\Gamma^{3,19}$ , so that there are some moduli for which (169) holds and the theory is supersymmetric. The gauge group of the 6D theory is given by

$$G = U(N_1) \times U(N_2) \times \cdots \times U(N_K) \times SO(M), \quad M = 32 - 2 \sum_a N_a. \quad (171)$$

The matter content can be determined from a simple index theory calculation and is given in the following table

Rep. (+ c.c.)	# hypermultiplets
$(N_a, N_b)$	$(-2 - (f_a + f_b)^2)$
$(N_a, \bar{N}_b)$	$(-2 - (f_a - f_b)^2)$
Antisym. $U(N_a)$	$(-2 - 4f_a^2)$
$(N_a, 2M)$	$(-2 - f_a^2)$
Neutral	20

It is easy to verify that anomalies cancel for any theory with this spectrum; for example, a small calculation shows that  $H - V + 29T = 273$ .

There are a few technical subtleties that we have glossed over here. Some of the  $U(1)$  factors may be anomalous and removed by a Stückelberg mechanism, as in the intersecting brane models discussed above [79]. Furthermore, there is a possibility of further massless states that may enhance the gauge spectrum when  $J \cdot f = 0, f^2 = -2$  corresponding to a rational curve on the K3 shrinking to a point [49].

Given that the spectrum and symmetry group of a 6D model constructed in this fashion is determined uniquely by the integers  $N_a$  and inner products of vectors  $f_a$  in the lattice  $L$ , it is natural to ask for what  $N_a, L$  models can be constructed. Given  $N_a, m_{ab}$  satisfying  $\sum_a N_a m_{aa} = -24$ , the criterion for existence of a model of this type is that there exist a lattice embedding

$$L(m) \hookrightarrow \Gamma^{3,19}. \quad (172)$$

It can be shown [49] using Nikulin's lattice embedding theorems that such an embedding is always possible for stack sizes and matrix  $m$  compatible with the 24-instanton condition (167). Furthermore, such an embedding is often unique, with degeneracies sometimes

associated with possible overlattice embeddings that would give rise to additional discrete structure in the 6D theory.

**Example:**

As a simple example, consider a model with gauge group

$$G = U(4) \times U(4) \times SO(16). \quad (173)$$

We have  $N_1 = N_2 = 4$ , so (167) becomes

$$4m_{11} + 4m_{22} = -24, \quad (174)$$

with  $m_{aa}$  an even integer for each  $a$ . The only solution (up to exchange of stacks) is

$$f_1 \cdot f_1 = -2, \quad f_2 \cdot f_2 = -4. \quad (175)$$

There are 5 distinct choices for  $m_{12}$  giving different theories; we consider the case where  $m_{12} = 0$ . From the table above we can compute the matter spectrum

$$\begin{aligned} \text{matter} = & 4 \times (\mathbf{4}, \bar{\mathbf{4}}, \mathbf{1}) + 4 \times (\mathbf{4}, \mathbf{4}, \mathbf{1}) + 6 \times (\mathbf{6}, \mathbf{1}, \mathbf{1}) + 14 \times (\mathbf{1}, \mathbf{6}, \mathbf{1}) \\ & + 2 \times (\mathbf{1}, \mathbf{4}, \mathbf{16}) + 20 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (176)$$

It is straightforward to check that this ( $T = 1$ ) model satisfies the anomaly equations and has anomaly lattice

$$\Lambda = \begin{pmatrix} 8 & 6 & 14 & -2 \\ 6 & 4 & 8 & 0 \\ 14 & 8 & 12 & 4 \\ -2 & 0 & 4 & -4 \end{pmatrix}. \quad (177)$$

Note that the two  $U(1)$  factors must be included in the number of vector multiplets for the condition (118) to be satisfied, and that both kinds of bifundamental hypermultiplets  $(\mathbf{4}, \mathbf{4})$  and  $(\mathbf{4}, \bar{\mathbf{4}})$  contribute to  $b_1 \cdot b_2$ . The matrix defining the anomaly lattice (177) is degenerate, which is expected since the dimension of this lattice must be  $T + 1 = 2$ . Indeed, the anomaly vectors can be chosen in the coordinate system (116) to be

$$-a = (2, 2) \quad b_1 = (1, 2) \quad (178)$$

$$b_2 = (1, 6) \quad b_3 = (1, -2). \quad (179)$$

These vectors correctly reproduce the matrix of inner products in (177). We will see in Section 4.8 how this structure can be used to easily map this model to the topological data for an F-theory compactification.

The general approach described here for compactification of the  $SO(32)$  theory on K3 can be repeated for the gauge group  $E_8 \times E_8$  with analogous results [107].

The models described here represent only a small slice of the full space of type I/heterotic compactifications on K3. More generally, the instanton structure can be nonabelian; the moduli space of vacua in this class is given by the moduli space of instantons on K3 with

instanton number 24. Tools for understanding instanton moduli spaces of this type were developed by Friedman, Morgan, and Witten [108]. By embedding nonabelian instantons with total instanton number 24 in a subgroup of the heterotic group, many other theories can be constructed [109, 110, 111]. For example, as pointed out by Kachru and Vafa, putting all the instantons in an  $SU(2)$  subgroup of one of the factors of  $E_8 \times E_8$  gives a theory with gauge group

$$G = E_7 \times E_8 \quad (180)$$

and charged matter content

$$\text{matter} = 10 \times (\mathbf{56}, 1). \quad (181)$$

Using the group theory coefficients

$$E_7 : \quad A_{56} = 1, C_{56} = 1/24, A_{\text{adj}} = 3, C_{\text{adj}} = 1/6 \quad (182)$$

$$E_8 : \quad A_{\text{adj}} = 1, C_{\text{adj}} = 1/100 \quad (183)$$

the anomaly lattice for this theory is

$$\Lambda = \begin{pmatrix} 8 & -14 & 10 \\ -14 & 12 & 0 \\ 10 & 0 & -12 \end{pmatrix}. \quad (184)$$

In the  $T = 1$  basis (116) we have

$$b_7 = (1, 6), \quad b_8 = (1, -6). \quad (185)$$

The continuous moduli space of theories including nonabelian instantons connects many of the models that appear with distinct gauge groups in the preceding analysis. Generically, the bundle structure will break the  $SO(32)$  gauge group to  $SO(8)$ . The different models described above are connected by Higgsing to a generic model, moving in moduli space, and un-Higgsing to restore a different gauge group. This connects part of the moduli space of 6D theories; we describe a more extensive connectivity of this moduli space in the context of F-theory in Section 4.7.

Note that while the magnetized brane models just described are morally “T-dual” to the intersecting models described in the previous section, the specific models arising from intersecting brane models of IIB on K3 do not appear in the magnetized brane picture. For example, the model from (152) with gauge group  $SU(8)$  does not arise as a magnetized brane model. This is because the orbifold quotient needed to realize K3 breaks the naive T-duality associated with  $T^4$ . Further structure is needed to make the connection between these classes of models more precise, some of which we describe briefly in the following section.

## 4.6 Other constructions of 6D type I/heterotic vacua

A wide range of other 6D models have been constructed from string theory, using a variety of techniques related to the heterotic/type I constructions described in the previous section.

We mention briefly some of these approaches and models to give a sense of the range of possibilities. A more comprehensive review of the range of models known can be found in the review by Ibáñez and Uranga [112]. As we discuss in the following section, F-theory provides a powerful unifying framework that seems to encompass all of the vacua constructed from these other approaches. The F-theory description of each of these models can be determined directly from the anomaly structure, as we discuss in Section 4.8. There are other types of 6D models related to the type I compactification on K3 that are constructed by generalizing the geometry of K3 to a “non-geometric space” [113]. These models also can be associated with F-theory constructions.

One class of models was constructed by Bianchi and Sagnotti based on combining the  $\mathbb{Z}_2$  orientifold action in the IIB theory with a space-time orbifold action giving  $T^4/\mathbb{Z}_2$  [114, 115, 116, 117]. One model of this type that was considered in more detail by Gimon and Polchinski gives a gauge group

$$G = SU(16) \times SU(16), \quad (186)$$

with matter content

$$\text{matter} = 2 \times (\mathbf{120}, \mathbf{1}) + 2 \times (\mathbf{1}, \mathbf{120}) + 1 \times (\mathbf{1}, \mathbf{1}). \quad (187)$$

For each gauge group factor the anomaly cancellation in this model follows as for the example with matter (127). This kind of type I model on K3 orbifolds can also give rise to theories with additional tensor multiplets [118, 119, 120].

A variety of heterotic compactifications on various toroidal orbifold limits of K3 of the form  $T^4/\mathbb{Z}_k$  were constructed by Erler and others [78, 121]. Tables of some of these models can be found in the work of Aldazabal *et al.* [122], and include models such as one with

$$G = SU(16), \quad \text{matter} = 2 \times \mathbf{120} + 16 \times \mathbf{16}. \quad (188)$$

Note that this is the model considered in (129) with the maximum possible value of  $N$ . Many heterotic orbifold models were considered by Honecker and Trapletti [79, 107] and connected to smooth K3 constructions.

From the heterotic point of view, more exotic theories with additional tensors can arise when instantons shrink to a point. This corresponds to a limit where the 5-brane charge encoded in the instanton congeals into a localized 5-brane and can separate from the ambient gauge theory [99]. The 5-brane world-volume theory carries a self-dual two-form field that corresponds to an additional tensor in the 6D theory. By shrinking  $k$  coincident instantons, a group  $Sp(k)$  arises on the 5-brane world-volume. The gauge groups can become even more exotic when the instantons shrink to a point with an orbifold singularity. A variety of models of this type have been constructed [123, 124, 125]. An extreme case of the kind of large gauge group that can arise was identified by Aspinwall, Gross and Morrison [126] in a theory with 192 vector multiplets and

$$G = E_8^{17} \times F_4^{16} \times G_2^{32} \times SU(2)^{32}. \quad (189)$$

A further complication that can arise in these heterotic constructions is the appearance of bundles without vector structure [127, 128]. This refers to a situation where an instanton attached to an orbifold singularity describes a  $\text{Spin}(32)/\mathbb{Z}_2$  bundle but not an  $SO(32)$  bundle.

From the heterotic/type I point of view, the wide range of possible models and underlying physical mechanisms, including all possible numbers of tensor fields, seems difficult to connect into a single systematic framework. We now turn to the point of view of F-theory, which provides a unifying perspective on the complicated network of possible 6D theories.

## 4.7 Six-dimensional vacua of F-theory

We now consider F-theory compactifications to six dimensions. The story is very similar in spirit to the description of eight-dimensional F-theory vacua in Section 3.4, though the details are more complicated. We begin by developing the structure of F-theory further to incorporate compactification on elliptically fibered Calabi-Yau threefolds. We then show how the anomaly data from any 6D theory can be used to identify the topological data for an F-theory realization of the theory, if one exists. This brings us to a point where we can discuss the global structure of the space of 6D  $\mathcal{N} = 1$  supergravity theories.

For a compactification of F-theory to six dimensions, we consider a Calabi-Yau threefold  $X$  that is elliptically fibered over a complex surface  $B$ . The canonical class of the base is  $K = c_1(T^*) \in H^2(B)$ , with a description in local complex coordinates  $\sim dz_1 \wedge dz_2$ . As in the case of a surface elliptically fibered over  $\mathbb{P}^1$ , there is a Weierstrass description of the elliptic fibration in terms of local functions  $f, g$  on the base

$$-y^2 + x^3 + f(s, t)x + g(s, t) = 0, \quad (190)$$

where  $s, t$  are local coordinates on the base. The functions  $f, g$  are given globally by sections of  $\mathcal{O}(-4K)$  and  $\mathcal{O}(-6K)$ , as in the 8D story. Again, codimension one singularities in the fibration give rise to the gauge group of the 6D theory. These are understood through the Kodaira classification as in 8D. In 6D theories, however, there are also codimension two singularities. These give rise to matter in the 6D theory. While some of the simplest kinds of codimension two singularities, giving rise to the simplest types of matter, are well understood, the mathematical description of codimension two singularities is as yet incomplete.

One approach to systematically constructing many elliptically-fibered Calabi-Yau geometries for F-theory compactifications is through the use of toric geometry [129, 130, 131, 132]. We do not discuss this approach in detail here, but it provides a concrete set of tools for explicitly constructing and analyzing a broad subset of the space of elliptically-fibered spaces used in F-theory constructions.

### 4.7.1 Codimension one singularities and gauge groups

The discriminant locus where the fibration is singular is given by the set of points where  $\Delta = 4f^3 + 27g^2 = 0$ . This is generically a codimension one locus on the base, which can be characterized as a divisor

$$\sum_i n_i H_i, \quad n_i \in \mathbb{Z} \quad (191)$$

where  $H_i$  are irreducible algebraic hypersurfaces on the base  $B$ . The divisor class of the discriminant locus is again given through the Kodaira condition for the total space to be Calabi-Yau

$$-12K = \Delta. \quad (192)$$

This divisor is *effective*, meaning that the coefficients  $n_i$  in an expansion of the form (191) satisfy  $n_i \geq 0$ . The irreducible components of  $\Delta$  can give rise to nonabelian gauge group factors, through the Kodaira classification in Table 1. We denote by  $\xi_i$  the irreducible components associated with nonabelian factors and by  $\nu_i$  the corresponding multiplicity (*i.e.* the degree of  $\Delta$  along  $\xi_i$ ). For example, an  $A_{N-1}$  singularity on a divisor class  $\xi$  has multiplicity  $\nu = N$ , while an  $E_8$  singularity has multiplicity  $\nu = 10$ . Each  $\xi_i$  is associated with an algebraic curve on the surface  $B$ . In the IIB picture, multiple 7-branes are wrapped on the curves  $\xi_i$ . There is a residual part of the discriminant locus, which we denote by  $Y$ , that is not sufficiently singular to give rise to nonabelian gauge factors. The total discriminant locus can thus be written

$$-12K = \Delta = \sum_i \nu_i \xi_i + Y, \quad (193)$$

where  $\xi_i$  gives rise to the nonabelian gauge group factor

$$\xi_i \rightarrow G_i. \quad (194)$$

The semisimple part of the gauge group for the 6D theory is then given by

$$G = G_1 \times \cdots \times G_k / \Gamma \quad (195)$$

where  $\Gamma$  is a discrete group. The abelian part of the gauge group arises in a more subtle fashion from the global structure of the elliptic fibration, as in 8D. More precisely, the rank of the abelian part of the gauge group is given by the dimension of the *Mordell-Weil* group of rational sections of the fibration [54, 133]. The structure of the discrete group  $\Gamma$  is determined by the torsion part of the Mordell-Weil group [134]. In terms of the total space  $X$  of the Calabi-Yau threefold, the total rank  $r$  of the gauge group, including both abelian and nonabelian factors, is given by [45]

$$r = h_{1,1}(X) - h_{1,1}(B) - 1. \quad (196)$$

While this is easy to compute when the geometry of the threefold is known, it is less straightforward to compute the abelian part of the gauge group given only the local characterization of the fibration, and requires either the explicit Weierstrass model or a detailed characterization of the codimension one and two components of the discriminant locus.

A further complication that can arise in 6D F-theory compactifications is the appearance of non-simply-laced groups outside the A-D-E classification [135, 136, 137]. This can occur when the set of singular curves associated with a codimension one singularity undergoes a monodromy around a nontrivial cycle in the base, so that the associated Dynkin diagram is mapped into itself in a nontrivial fashion (*i.e.*, through an outer automorphism). For example, an  $A_7$  singularity that undergoes a monodromy reflecting the Dynkin diagram

from one end to the other contracts the Dynkin diagram through the reflection, giving rise to a group  $Sp(4)$  associated with a  $C_4$  singularity. Treating such groups in F-theory requires a slightly more careful analysis of the singularity structure on the dimension one discriminant locus, for which a systematic procedure known as the “Tate algorithm” is helpful [138].

#### 4.7.2 Codimension two singularities and matter

Matter in the 6D theories is produced from codimension two singularities in the elliptic fibration. In general, at codimension two points in the base  $B$ , the singularity structure of the fibration can worsen. This can lead to additional points in the base where the total space of the fibration is singular and must be blown up into two-cycles to give a smooth Calabi-Yau threefold. These two-cycles represent additional matter fields in the theory. In general, the codimension two singularity at a point  $x$  in the base gives an enhancement of the Kodaira type of the singularity along one or more codimension one curves  $\xi_i$  passing through the point  $x$ . In the simplest kind of codimension two singularity, as described by Katz and Vafa [139], the singularity at  $x$  is resolved by blowing up a Kodaira type associated with a group of one rank higher than the codimension one singularity/singularities intersecting that point. In this case the matter content is found by decomposing the adjoint of the larger group into the gauge factor(s) associated with the codimension one locus. In other cases, the codimension two singularity can give rise to matter in a more complicated fashion [140, 141]. One general class of codimension two singularities has been analyzed by Miranda [142], though more general classes of singularities are possible that have not yet been completely classified.

##### Example:

Consider a local singularity of type  $A_{n-1}$  along a curve  $\xi_1$ , which we can take to be on the line  $t = 0$ , with  $\Delta \sim t^n$ . If there is also a codimension one singularity of type  $A_{m-1}$  along the curve  $\xi_2$  given by the line  $s = 0$ , with  $\Delta \sim s^m$ , then at the origin where the curves intersect, the total singularity type is  $A_{n+m-1}$ , since  $\text{ord } (\Delta) = n + m$ . The two components of the codimension one singularity locus along  $\xi_1, \xi_2$  give a group  $SU(n) \times SU(m)$ . The enhanced singularity at the origin is associated with the adjoint of  $SU(n + m)$  (although this is not a part of the gauge symmetry). Decomposing the adjoint representation of  $SU(n + m)$  gives a bifundamental field in the  $(n, \bar{m}) + (\bar{n}, m)$  representation of the two gauge group factors. This is the F-theory realization of the bifundamental strings arising from intersecting branes.

The mechanism by which matter arises in F-theory is thus related to the standard mechanism realizing matter from strings connecting intersecting branes. But in F-theory there is a much wider range of possible matter structures [139, 143, 141]. For example, a trifundamental matter field can be produced by a local enhancement of an  $A_1 \times A_2 \times A_4$  singularity to  $E_8$

$$\bullet\bullet\bullet\bullet\bullet \Rightarrow (\square_3, \square_2, \square_4)$$

In addition to matter from localized singularities, there are global contributions to matter in F-theory. In particular, a codimension one singularity locus associated with a gauge

group factor  $G_i$  that is wrapped on a curve of genus  $g$  in the base gives rise to  $g$  adjoint representations of  $G_i$ .

#### 4.7.3 Bases for elliptically fibered Calabi-Yau threefolds

We now consider the question of which bases  $B$  can be used for an F-theory compactification on an elliptically fibered Calabi-Yau threefold. A full discussion of the mathematics underlying the answer to this question is beyond the scope of these lectures, but we summarize the main points here.

The only bases that can support an elliptically fibered Calabi-Yau threefold are given by the following set of spaces [144, 145, 45]:

$T^4$ , K3: These spaces both have  $K = 0$ . So there is no discriminant locus and the fibration is trivial. Thus, these are simple IIB compactifications, and the general structure of F-theory is not needed. These models give theories with enhanced supersymmetry. Models with enhanced supersymmetry can also arise from compactification on hyperelliptic surfaces or surfaces of the form  $(T^2 \times \mathbb{P}^1)/G$  with  $G$  a discrete group; we do not consider any of these models further here.

Enriques surface: This is an orbifold of K3 with no fixed points. The space has  $-12K = 0$ , with  $K$  a torsion class. So there are no nonabelian gauge group factors on the discriminant locus. F-theory can nonetheless be compactified on the Enriques surface [146]. The dyon charge lattice for the theory is [147]  $\Gamma^{1,9}$ .

$\mathbb{P}^2$ : The two-dimensional complex projective space  $\mathbb{P}^2$  has  $H_2(\mathbb{P}^2) = \mathbb{Z}$ , with  $K = -3H$ , where  $H$  is the hyperplane divisor generating  $H_2(\mathbb{P}^2)$ .

$\mathbb{F}_m$ : Hirzebruch surfaces with  $m \leq 12$  (described in more detail below).

Blow-ups of  $\mathbb{P}^2, \mathbb{F}_m$  at one or more points: Blowing up one or more points on these spaces gives surfaces with increasingly large  $H_2(B, \mathbb{Z})$ .

It is also possible that there may be consistent F-theory compactifications on bases with orbifold singularities, whose resolution gives one of the spaces above [45]; understanding such compactifications is an interesting open problem.

The fact that no other surfaces besides those listed above can be used as bases for an elliptically fibered Calabi-Yau manifold follows from results in *minimal surface theory*. Basically, the idea of minimal surface theory is that on any surface which contains a divisor class describing a curve  $C$  satisfying  $C \cdot K = C \cdot C = -1$ , the curve  $C$  can be blown down to give a simpler smooth surface [148]. Minimal surfaces are those which admit no further curves which can be blown down in this fashion. The only minimal surfaces which can be bases for elliptically fibered Calabi-Yau manifolds are those listed above.

It will be helpful to have on hand some more details of the structure of the Hirzebruch surfaces  $\mathbb{F}_m$ . These surfaces can be described as  $\mathbb{P}^1$  bundles over  $\mathbb{P}^1$ ; the surface  $\mathbb{F}_m$  is essentially equivalent to the line bundle over  $\mathbb{P}^1$  with first Chern class  $c_1 = -m$  with all fibers compactified to  $\mathbb{P}^1$ . For example,

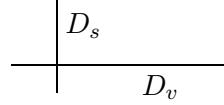
$$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1. \tag{197}$$

The surface  $\mathbb{F}_1$  is also described by blowing up  $\mathbb{P}^2$  at a point. The surface  $\mathbb{F}_2$  is given by compactifying  $T^*\mathbb{P}^1$  on each fiber.

A basis for the linear space of divisors of  $\mathbb{F}_m$  is given by the divisors  $D_v, D_s$ , where  $D_v$  is a section of the  $\mathbb{P}^1$  fibration with  $D_v^2 = -m$ , and  $D_s$  is a fiber of the fibration. The intersection matrix in this basis is given by

$$\begin{pmatrix} D_v \cdot D_v & D_v \cdot D_s \\ D_s \cdot D_v & D_s \cdot D_s \end{pmatrix} = \begin{pmatrix} -m & 1 \\ 1 & 0 \end{pmatrix}. \quad (198)$$

Geometrically we can picture  $D_v$  as a horizontal line depicting the section and  $D_s$  as a vertical line depicting the fiber



The irreducible effective divisor classes in  $\mathbb{F}_m$  are given by

$$D_v, aD_v + bD_s, \quad b \geq ma. \quad (199)$$

The canonical class on  $\mathbb{F}_m$  is

$$-K = 2D_v + (2 + m)D_s. \quad (200)$$

It is easy to check that for any  $m$ ,  $K^2 = 8$  on  $\mathbb{F}_m$ .

From the divisor structure of  $\mathbb{F}_m$  we can see that  $\mathbb{F}_m$  can only be the base for an elliptically fibered Calabi-Yau threefold with singularities that are in the Kodaira table when  $m \leq 12$ . From  $f = -4K$ , we see that for  $m > 2$ ,  $f$  must have an irreducible component proportional to  $D_v$ , since from eq. (199) any irreducible component other than  $D_v$  must have a coefficient of  $D_s$  at least  $m$  times the coefficient of  $D_v$ . Writing

$$f = x_f D_v + y = x_f D_v + (AD_v + BD_s) = 8D_v + (8 + 4m)D_s \quad (201)$$

where  $y$  is a sum of irreducible components other than  $D_v$ , we have  $B = 8 + 4m \geq mA$ , so  $A \leq 4 + 8/m$ . It follows that

$$x_f \geq 4 - 8/m. \quad (202)$$

A similar analysis for  $g = -6K$  and  $\Delta = -12K$  gives

$$x_g \geq 6 - 12/m, \quad x_\Delta \geq 12 - 24/m. \quad (203)$$

For  $m > 12$ , the divisor class  $D_v$  thus carries a singularity of degrees  $\deg(f) \geq 4$ ,  $\deg(g) > 5$ ,  $\deg(\Delta) > 10$ . Since  $D_v^2$  is negative, there are no deformations of this divisor class, so there is a singularity worse than the Kodaira classification allows on this locus at  $m > 12$ .

It is easy to verify that for  $\mathbb{F}_m$ , the lattice  $\Gamma = H_2(B, \mathbb{Z})$  spanned by  $D_v, D_s$  can be written in an appropriate basis as

$$\Gamma = \Gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, m \text{ even}, \quad \Gamma = \Gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, m \text{ odd}. \quad (204)$$

In fact, all the even Hirzebruch surfaces  $\mathbb{F}_{2m}$  are topologically equivalent to one another, as are all the odd surfaces  $\mathbb{F}_{2m+1}$ . The difference between two even or odd Hirzebruch surfaces with distinct  $m$  lies in the complex structure, which is associated with a different set of effective irreducible divisors for each  $m$ .

Blowing up a generic point on a surface  $B$  adds an additional divisor class  $E$  (the exceptional divisor), giving a new surface  $\tilde{B}$  with

$$E^2 = -1, \quad \tilde{K} \cdot E = -1. \quad (205)$$

In the new surface,  $\tilde{K} = K + E$ . The dimension of the second homology group increases by 1 with the blow-up so that  $h_{1,1}(\tilde{B}) = h_{1,1}(B) + 1$ . For example, if we blow up a generic point on  $\mathbb{F}_1$  the resulting surface is the *del Pezzo* surface  $dP_2$ . The intersection form on  $dP_2$  is given by adding a new dimension corresponding to the orthogonal generator  $E$  to  $\Gamma_1$  from eq. (204), and taking  $K_{dP_2} = K_{\mathbb{F}_1} + E$ ,

$$\Gamma_{dP_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (206)$$

in a basis where  $K_{dP_2} = (-3, 1, 1)$ . Blowing up a generic point on  $dP_k$  gives  $dP_{k+1}$ . Blowing up non-generic points gives a more complicated tree of resulting surfaces. Each blow-up, however, has the effect of reducing  $K^2$  by 1 through the addition of the new exceptional divisor. In general, for any base we have

$$K^2 = 10 - h_{1,1}(B). \quad (207)$$

The number of tensor multiplets in the 6D supergravity theory associated with F-theory on an elliptic fibration over the base  $B$  is given by

$$T = h_{1,1}(B) - 1. \quad (208)$$

The scalar fields in the  $T$  tensor multiplets correspond to the relative Kähler (volume) moduli for the  $h_{1,1}$  two-cycles in the base, leaving out the overall volume modulus of  $B$ , which is controlled by a scalar hypermultiplet. Thus, fibrations over  $\mathbb{P}^2$  correspond to theories with no tensor multiplets, fibrations over  $\mathbb{F}_m$  give theories with one tensor multiplet, and each additional blow-up of the base gives an additional tensor multiplet. Combining (207) and (208) gives

$$K^2 = 9 - T = 10 - h_{1,1}(B). \quad (209)$$

The mathematical connection between a Calabi-Yau elliptically fibered over a base  $B$  and a Calabi-Yau fibered over a base  $\tilde{B}$  realized by blowing up  $B$  at a point is mirrored in the physics of the associated supergravity theories. In the supergravity theory, such a transition is realized by a phase transition in the theory in which a single tensor multiplet is exchanged for 29 scalar multiplets [149, 150, 45]. Starting from the theory on  $\tilde{B}$  with the larger value of  $T$ , this transition can be described by a limit in which a string becomes tensionless as

$j \cdot x \rightarrow 0$  for some  $x \in \Gamma$ . From the F-theory point of view, the transition from  $B$  to  $\tilde{B}$  can be seen as arising from a tuning of the Weierstrass parameters for an elliptic fibration over  $B$  so that a singularity arises that is worse than any singularity in the Kodaira Table. For example, at a codimension two singularity where  $\deg f = 4$ ,  $\deg g = 6$ , and  $\deg \Delta = 12$ , the singularity must be resolved by blowing up a point in the base. These transitions connect F-theory models with different bases  $B$  and different numbers of tensor multiplets  $T$ . We discuss this connectivity in the space of supergravity theories further below in Section 4.9.2.

We now consider a simple example of an F-theory compactification on an elliptically fibered threefold, focusing on the topological data. As additional examples, many of the other theories constructed through other methods are mapped into F-theory in Section 4.8.

### Example: $SU(N)$ on $\mathbb{F}_2$

This is a 6D supergravity theory with  $T = 1$ . If the  $SU(N)$  is realized by  $N$  D7-branes on the divisor class

$$\xi = D_v \quad (210)$$

then we have

$$-12K = 24D_v + 48D_s = ND_v + Y, \quad (211)$$

where  $Y = (24 - N)D_v + 48D_s$ . Matter in the fundamental representation of the  $SU(N)$  arises at intersections between  $\xi = D_v$  and  $Y$ . The number of such intersections is

$$\xi \cdot Y = D_v \cdot [(24 - N)D_v + 48D_s] = 48 - 2(24 - N) = 2N, \quad (212)$$

so the theory has  $2N$  matter fields in the fundamental representation of  $SU(N)$ . It is easy to check that this is an anomaly-free spectrum. An explicit Weierstrass model for this F-theory compactification for  $N \leq 14$  has been identified [151].

To fit the set of models described through other constructions into the context of F-theory, it is useful to exploit the close correspondence between the anomaly structure of 6D supergravity and the structure of F-theory compactifications, to which we now turn.

## 4.8 Mapping 6D supergravities to F-theory

As discussed in Section 4.2, every 6D supergravity theory contains a signature  $(1, T)$  lattice  $\Gamma$  of dyonic string charges that couple to the (anti-)self-dual fields  $B^\pm$  in the theory. From the F-theory point of view these dyonic strings are realized by D3-branes wrapping cycles in  $H_2(B, \mathbb{Z})$  of the base. Thus,

$$\Gamma = H_2(B, \mathbb{Z}) \quad (213)$$

with inner product given by the intersection form on  $B$ . In F-theory the self-duality of the lattice  $\Gamma$  follows immediately from Poincaré duality.

We now explicitly describe how the anomaly lattice  $\Lambda$  of a 6D supergravity theory is mapped into the charge lattice  $\Gamma$  for theories with an F-theory realization [151, 77]. The divisor classes associated with nonabelian gauge group factors in a 6D F-theory compactification live in  $\Gamma$  since

$$\xi_i \in H_2(B, \mathbb{Z}). \quad (214)$$

The nonabelian gauge group factor  $G_i$  comes from a stack of 7-branes wrapped on the cycle  $\xi_i$ . The corresponding element  $b_i$  in the anomaly lattice comes from a gauge dyonic string in space-time associated with an instanton in  $G_i$ . In F-theory this instanton gives a 3-brane wrapped on  $\xi_i$  within the 7-brane world-volume. Thus, we can associate the element  $b_i$  of the anomaly lattice with  $\xi_i$

$$b_i \rightarrow \xi_i. \quad (215)$$

From the anomaly condition (120) and the condition (209) we have  $a^2 = K^2 = 9 - T$ . This suggests that

$$a \rightarrow K. \quad (216)$$

Indeed, both (215) and (216) can be checked by computing the intersections in the F-theory picture and the corresponding anomaly lattice [76, 140]

$$-a \cdot b = -K \cdot \xi_i \quad (217)$$

$$b_i \cdot b_j = \xi_i \cdot \xi_j \quad (218)$$

This gives a clear picture of how the topological data needed for an F-theory compactification can be identified from the structure of the 6D supergravity spectrum and anomaly lattice. Given a 6D theory, the number of tensor multiplets  $T$  determines  $h_{1,1}(B)$  through (208). There must then exist a lattice embedding

$$\Lambda \hookrightarrow \Gamma \quad (219)$$

that maps  $a, b_i \rightarrow K, \xi_i$  as in (216), (215). This map takes the  $(1, T)$  vector  $j$  of scalars in the tensor multiplets to the Kähler moduli of the F-theory compactification

$$j \rightarrow J. \quad (220)$$

The mapping (219) is not necessarily uniquely defined given only the nonabelian gauge group and matter content of the 6D theory. There may be multiple possible surfaces  $B$  giving compatible F-theory compactifications, and for a given surface the lattice map may not be uniquely defined, although in many cases it can be shown that the embedding is unique up to automorphisms of  $\Gamma$  using theorems of Nikulin. To uniquely determine the F-theory model corresponding to a given 6D supergravity, further information may be needed. Knowing the dyon charge lattice of the low-energy theory and having an explicit description of the space of possible Kähler moduli encoded in  $j$  (*i.e.*, the Kähler cone), for example, is sufficient to uniquely determine the intersection form on  $B$ , along with the set of effective divisors. This information uniquely determines the F-theory realization. Knowing the  $U(1)$  content of the theory can also in principle help in determining the topology of the F-theory geometry. In many simple cases, however, as we see in examples below, the nonabelian gauge group and matter content are already sufficient to uniquely determine the corresponding F-theory realization.

In the case  $T = 1$ , the map (219) can be written explicitly in terms of the divisors  $D_v, D_s$  on  $\mathbb{F}_m$  satisfying (198). The F-theory divisor corresponding to an anomaly vector  $b$  is given

by [151]

$$b = \frac{1}{2}(\alpha, \tilde{\alpha}) \rightarrow \xi = \frac{\alpha}{2}(D_v + \frac{m}{2}D_s) + \frac{\tilde{\alpha}}{2}D_s \quad (221)$$

where we have expressed  $b$  in terms of  $\alpha, \tilde{\alpha}$  as in (116). This map is also compatible with the corresponding expression for  $a$

$$-a = (2, 2) \rightarrow -K = 2D_v + (m+2)D_s. \quad (222)$$

Note that the anomaly formalism is in this case invariant under exchange of  $\alpha, \tilde{\alpha}$ . This can in some cases give distinct realizations of a given 6D model (which are often related through duality, such as for models on  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ , where exchange of  $\alpha, \tilde{\alpha}$  corresponds to an exchange of  $D_v$  and  $D_s$ .)

Dualities between heterotic and F-theory models have been worked out in many special cases and classes of examples in various parts of the literature; this connection is explored in many of the papers referenced in the discussion of heterotic orbifold models, in particular in the work of Friedman, Morgan and Witten [108], as well as in much of the F-theory literature. The map (219) gives a simple unified way of identifying such dualities in terms of the discrete data of the 6D supergravity theory and the corresponding F-theory topology. We now consider a number of explicit examples of this map.

**Example:**

Consider again the theory with gauge group  $G = SU(N)$  and  $2N$  fundamental matter fields, with  $T = 1$ . A straightforward calculation gives

$$b = \frac{1}{2}(\alpha, \tilde{\alpha}) = (1, -1) \rightarrow \xi = D_v + (m/2 - 1)D_s. \quad (223)$$

The map will only give an integral effective divisor for  $m = 2$ . This gives  $\xi = D_v$  on  $\mathbb{F}_2$ , as described in (210).

**Example:**

Consider the intersecting brane model example (152) with gauge group  $U(8)$ , 8 matter fields in the antisymmetric two-index representation and one adjoint matter field. We saw from (155) that the anomaly lattice is generated by a single vector  $b = -a$ . Using (219), we see that this model maps to  $\mathbb{F}_0$  with

$$b = (2, 2) \rightarrow \xi = 2D_v + 2D_s. \quad (224)$$

The genus of this curve is given by

$$(K + \xi) \cdot \xi = (a + b) \cdot b = 0 = 2g - 2. \quad (225)$$

So the  $SU(8)$  group lives on a discriminant locus curve of genus one, as expected given the one adjoint representation.

**Example:**

Now consider the example of a type I/heterotic K3 compactification with gauge group (173) and matter content (176). From the form of the anomaly vector  $b_3 = (1, -2)$  associated

with the gauge group factor  $SO(16)$ , we see that the only possible value for  $m$  that gives an effective divisor for the image of  $b_3$  under (221) is  $m = 4$ . The divisor classes associated with the different gauge group factors are then

$$\begin{aligned} b_1 &= (1, 2) \rightarrow D_u = D_v + 4D_s & (226) \\ b_2 &= (1, 6) \rightarrow D_v + 8D_s \\ b_3 &= (1, -2) \rightarrow D_v \end{aligned}$$

So we expect a  $D_8$  singularity on the divisor  $D_v$ , and  $A_3$  singularities on  $D_v + 4D_s$  and  $D_v + 8D_s$ . This general structure is typical for smooth heterotic compactifications of the  $SO(32)$  theory on K3; the F-theory realization of such models is always associated with the base  $\mathbb{F}_4$ , and the residual  $SO(M)$  factor resides on  $D_v$  [45].

**Example:**  $E_7 \times E_8$  on  $\mathbb{F}_{12}$

We discussed above the model (180) with gauge group  $E_7 \times E_8$ . From the vector  $b_8 = (1, -6)$  in (185), we see that (219) only gives an effective divisor for  $m = 12$ . The  $E_8$  and  $E_7$  loci are then  $D_v, D_u = D_v + 12D_s$  respectively. There is no bifundamental matter, as expected, since  $D_v \cdot D_u = 0$ . This is an example of a heterotic  $E_8 \times E_8$  vacuum with all instantons in one  $E_8$ . For general heterotic  $E_8 \times E_8$  compactifications, the resulting F-theory model has a base  $\mathbb{F}_m$  where  $12 \pm m$  of the instantons are embedded in each  $E_8$  factor [45].

**Example:** Gimon-Polchinski model

Consider now the Gimon-Polchinski model (186), with gauge group  $G = SU(16) \times SU(16)$ , two matter fields transforming under the antisymmetric **120** for each gauge group factor, and one bifundamental matter field. It is easy to compute that the anomaly matrix for this model is

$$\Lambda = \begin{pmatrix} 8 & -2 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}. \quad (227)$$

It follows that in the canonical  $T = 1$  basis (116),

$$b_1 = (1, 0), \quad b_2 = (0, 1). \quad (228)$$

This theory must therefore be realized through an F-theory compactification on  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ , with divisor classes  $b_1 \rightarrow D_v, b_2 \rightarrow D_s$ . This F-theory realization of the Gimon-Polchinski model has been explored in some detail as a useful example for studying the heterotic-F-theory duality, in particular clarifying the role of instantons with  $Spin(32)/\mathbb{Z}_2$  structure but not  $SO(32)$  vector structure [127, 128, 152].

The reader may find it illuminating to work out a few further examples. As far as the author is aware all known 6D supergravity theories that arise from any kind of string compactification admit an embedding into F-theory through a map of the form (219), at least at the level of the discrete data characterizing the gauge group and matter content of the theory.

## 4.9 Global structure of the space of 6D $\mathcal{N} = 1$ supergravities

### 4.9.1 Consistency conditions, and matching $\mathcal{G}$ to $\mathcal{V}$

In Section 4.2 we summarized the known constraints on the space  $\mathcal{G}^{6D, \mathcal{N}=1}$  (which we refer to simply as  $\mathcal{G}$  for the remainder of this section). For  $T < 9$  there are a finite set of possible combinations of nonabelian gauge groups and matter content compatible with anomaly cancellation conditions and proper-sign gauge kinetic terms. For  $T \geq 9$  there are infinite families of models that satisfy the known consistency conditions. Consistent supergravity theories must furthermore satisfy the condition that the dyonic string charge lattice  $\Gamma$  is unimodular.

Given the finite constraint on the set of possible groups and matter for  $T < 9$ , it is possible in principle to enumerate all models consistent with the known consistency constraints. This can be done in practice more easily for smaller values of  $T$ . For  $T = 0$  the constraints are strongest, as the vectors in the anomaly lattice are just integers  $b \in \mathbb{Z}$ , and a complete classification of models is tractable [82]. For  $T = 1$  the space of possible combinations of groups and representations becomes larger; an exploration of part of this space of theories with some restriction on matter representations was initiated by Avramis and Kehagias [153]. Using the same methods that lead to the finiteness bounds, a complete classification of models has been carried out for gauge groups with  $SU(N)$  and certain matter representations [151], such a classification can be done for arbitrary gauge groups and matter. Incorporating the unimodularity constraint on the dyon charge lattice, such a systematic classification of models can in principle be continued up to  $T = 8$ .

Six-dimensional F-theory constructions, as described in Section 4.7, give a wide range of string vacua, and define a set  $\mathcal{V} = \mathcal{V}^{6D, \mathcal{N}=1}$  of known string vacuum constructions in 6D. For any 6D supergravity theory admitting an F-theory realization, the anomaly lattice of the 6D theory is mapped into the topological structure of the F-theory construction through the map (219). This provides a framework for identifying general features of models lying in the set  $\mathcal{G} \setminus \mathcal{V}$ , which satisfy all known consistency conditions and yet have no known string theory construction. Basically, we just need to determine which models in  $\mathcal{G}$  do not lead to acceptable data for an F-theory model. There are a number of specific ways in which the map (219) can fail to take the anomaly lattice of an apparently-consistent theory to valid topological data for an F-theory compactification. We list a number of these possible failure modes and comment on each.

- Unimodular embedding.

A necessary condition for the existence of an embedding in F-theory through (219) is the existence of an embedding of  $\Lambda$  in a unimodular lattice. As discussed at the end of Section 4.2, however, it has been shown [87] that every consistent 6D theory has a unimodular charge lattice, of which  $\Lambda$  is a sublattice. Thus, every consistent theory has an anomaly lattice that can be embedded in some unimodular lattice  $\Gamma$ .  $\Gamma$  can then be taken to be the homology lattice  $H_2(B, \mathbb{Z})$  of an F-theory base.

- Exotic matter fields.

There are a variety of 6D supergravity models that satisfy the known consistency condi-

tions and that have exotic matter representations not realized through known codimension two singularity types in F-theory [151, 77, 82]. As examples, consider the following theories (listing only charged matter for each theory)

$$T = 1, G = SU(8), \quad \text{matter} = \square + 3 \square\square + 2 \begin{smallmatrix} \square \\ \square \end{smallmatrix} + \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \quad (229)$$

$$T = 0, G = SU(4), \quad \text{matter} = 1 \times \begin{smallmatrix} \square \\ \square \end{smallmatrix} + 64 \times \square \quad (230)$$

$$T = 0, G = SU(8), \quad \text{matter} = 1 \times \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \quad (231)$$

Each of these theories contains a matter representation that cannot be realized using a standard codimension two F-theory singularity. The range of possible matter fields that may be realized using more exotic codimension two singularities is, however, quite large [141]. It may be possible to implement at least the first two of these models using more exotic types of codimension two singularity. One clue to this connection is given by a relationship between the group theory of matter representations and the corresponding F-theory geometry [82]. For any representation we can define the quantity

$$g_R = \frac{1}{12} (2C_R + B_R - A_R). \quad (232)$$

From the anomaly relations it is possible to show that for models with  $T = 0$  the set of representations transforming under a gauge factor  $G$  satisfies

$$\sum_R x_R g_R = \frac{(b-1)(b-2)}{2}. \quad (233)$$

In a  $T = 0$  F-theory realization,  $b$  is the degree of the polynomial defining the curve  $\xi$  on  $\mathbb{P}^2$  carrying the singularity associated with the gauge factor  $G$ . The quantity (233) is precisely the arithmetic genus of such a curve [50]. The arithmetic genus carries a contribution from the topological (geometric) genus as well as a contribution from singular points on the curve. So it is natural to expect that  $g_R$  encodes the arithmetic genus contribution of a singularity on the curve  $\xi$  carrying the group factor  $G$ . This correspondence works, for example, for the two-index symmetric representation of  $SU(N)$ , which has  $g_{\square\square} = 1$ , and is produced by an ordinary double point singularity contributing 1 to the arithmetic genus [76, 141]. A complete analysis of codimension two singularities in F-theory is a challenge for future research. Note, however, that some representations, such as the “box” representation of  $SU(8)$  appearing in (231) cannot appear in any known kind of F-theory model for other reasons; in the case of the  $SU(8)$  box representation, the theory violates the bound associated with the Kodaira condition (which is discussed further below). There are also constraints on matter representations that can be realized from the heterotic point of view [154, 155]. Understanding how the heterotic, F-theory, and supergravity constraints on allowed representations may be related is an interesting open problem.

- Weierstrass formulation

In the discussion here we have focused on the topological structure of the F-theory realization, in particular on the divisor classes supporting the singularities associated with a given nonabelian gauge group. For a complete F-theory realization, an explicit Weierstrass model is needed, where  $f, g$  are described explicitly as sections of the appropriate line bundles  $\mathcal{O}(-4K), \mathcal{O}(-6K)$ . In a variety of cases that have been studied, Weierstrass models can be found whenever a map of the form (219) can be found whose image gives divisor classes satisfying all the topological constraints from F-theory (including the positivity and Kodaira constraints formulated below). In cases where the topological conditions are satisfied, the number of degrees of freedom in the Weierstrass model that remain unfixed when the coefficients are tuned to achieve the desired gauge group precisely matches the number of uncharged scalar fields in the low-energy theory [151]. This led Kumar, Morrison, and the author to conjecture that the degrees of freedom will match in all cases, and that a Weierstrass model will exist whenever the topological constraints are satisfied. Finding a proof of this conjecture is left as an open problem for further research.

- **Positivity conditions and effective divisors**

There is a sign condition on the divisors appearing in the image of the map (219). The divisors  $\xi_i$  (associated with the curves where the 7-branes are wrapped) must be effective divisors; this is equivalent to the statement that there exists a moduli vector  $J$  such that

$$J \cdot \xi_i > 0 \quad \forall i. \quad (234)$$

We understand this condition physically in the 6D supergravity theory as the constraint that the gauge kinetic term for  $G_i$  must have the proper sign, *i.e.*  $j \cdot b_i > 0$ .

It is also the case for all F-theory models that  $-K$  is an effective divisor, so that  $J \cdot (-K) > 0$ . This condition in the supergravity theory states that

$$j \cdot a < 0. \quad (235)$$

This fixes the sign of the curvature-squared term in the 6D theory of the form  $(a \cdot j)R^2$ . It is not clear whether this sign constraint is necessary for consistency of the low-energy theory. It is possible that the sign of this term is fixed through a causality constraint analogous to those studied by Adams *et al.* [156]. A complete explanation of this constraint from the supergravity point of view is left as another open question for future work.

- **Kodaira constraint**

Related to the positivity conditions just mentioned, there is also a condition on the residual divisor locus  $Y = -12K - \sum_i \nu_i \xi_i$ , which states that this divisor is effective. This condition arises from the Kodaira constraint (193). In the 6D supergravity theory this constraint states that

$$j \cdot (-12a - \sum_i \nu_i b_i) > 0. \quad (236)$$

All of the known infinite families of models that satisfy the other macroscopic 6D constraints violate this ‘‘Kodaira bound’’. Thus, proving this bound would potentially reduce the space of possibly consistent combinations of  $T, G$ , and  $\mathcal{M}$  to a finite set  $\mathcal{G}$ . One possible route

to proving (236) would be to follow an approach along the lines described in Section 3.5. It may be that a careful analysis of world-sheet anomalies on the dyonic strings of a 6D supergravity theory will constrain the set of allowed models in such a way that (236) is necessary for consistency of the theory. Further study is needed, however, to see whether this speculation is borne out in practice.

#### 4.9.2 Connectivity and finiteness of the space of theories

Many, if not all, 6D supergravity theories with different gauge groups and matter content are connected through continuous deformations in the scalar moduli space. Moving in the moduli space can lead to a familiar Higgs type transition where a gauge group is broken, removing scalar fields and vector fields from the theory in such a way that the difference  $H - V$  is unchanged and the anomaly relation (118) remains valid. From the F-theory point of view this kind of transition arises from changing the coefficients in the Weierstrass functions  $f, g$  so that the structure of the discriminant locus changes. It is clear from the F-theory point of view that any two models associated with the same base  $B$  for the elliptic fibration are continuously connected through a deformation of the Weierstrass equation.

There are, however, further connections in the space of supergravity theories. As discussed in Section 4.7.3, supergravity theories with different numbers of tensor multiplets  $T$  can be connected through exotic phase transitions where a tensor multiplet  $T$  is exchanged for 29 hypermultiplets (with a possible additional change in the vector multiplet structure) in such a way that the anomaly relation (118) is preserved. Such a transition occurs in a limit of the moduli space where  $j \cdot x \rightarrow 0$  for some  $x \in \Gamma$  in the lattice of dyonic string charges. Thus, this transition is associated with a string becoming tensionless [149, 150, 45]. From the F-theory point of view this corresponds to a cycle in the base  $B$  shrinking to a point, *i.e.* to a blow-down (the opposite of a blow-up) of a rational curve in the surface. From the heterotic point of view, the transition from the theory with fewer tensor multiplets occurs when an instanton shrinks to a point, as mentioned above, producing a 5-brane carrying the additional tensor field. Since all smooth F-theory bases corresponding to theories with one 6D supersymmetry can be connected by a sequence of blow-up and blow-down operations, this suggests that the full space of 6D  $\mathcal{N} = 1$  supergravities is a single connected moduli space<sup>9</sup>. This would mean that there is really a single unified quantum theory of 6D supergravity since the entire moduli space is in principle visible in the structure of fluctuations of massless scalar fields around any given vacuum. This gives a simple and appealing picture of supergravity in six dimensions.

From the point of view of F-theory, the number of distinct bases  $B$ , as well as the number of combinations of  $T, G$ , and matter representations that can be realized must be finite. We summarize briefly the argument for finiteness [77]; the related fact that the number of topological classes of elliptically fibered Calabi-Yau threefolds (up to birational equivalence) is finite was proven by Gross [158]. From the minimal surface point of view, any F-theory model can be viewed as a Weierstrass model over either  $\mathbb{P}^2$  or one of the Hirzebruch

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<sup>9</sup>Extremal transitions associated with theories having the Enriques surface as base are identified in [157].

surfaces  $\mathbb{F}_m$ . If the number of tensor multiplets is greater than 1, then the Weierstrass model can be thought of as a Weierstrass model on  $\mathbb{F}_m$  with singularities tuned to require one or more blow-ups of points on  $B$ , as discussed in Section 4.7.3. Each different combination of base, gauge group, and matter content, then, corresponds to a locally closed set in the finite-dimensional space of Weierstrass models. (A locally closed set is an intersection of a closed set with an open set. Each condition on the discriminant locus gives a closed set, with each further condition giving an enhanced gauge group or further blow-up giving a smaller closed set within the original set.) A fundamental theorem from commutative algebra and algebraic geometry, the Hilbert basis theorem, guarantees that there are only a finite number of distinct such locally closed sets. Heuristically this follows from the observation that each gauge group enhancement or blow-up requires tuning a finite number of parameters, and the total number of parameters is finite and equal to the number of neutral scalar fields in the generic model, 273 on  $\mathbb{P}^2$  and 244 on  $\mathbb{F}_m$ . Since the known constraints on  $\mathcal{G}$  give an infinite set of distinct models, at present, the “apparent swampland” of 6D  $\mathcal{N} = 1$  theories is infinite. Finding an argument for the Kodaira constraint as a consistency condition for any supergravity theory could eliminate the infinite number of apparently consistent models that are not realized in F-theory.

While the set of allowed F-theory models is provably finite, the precise extent of the set of F-theory compactifications has not been determined. For example, the maximum number of tensor multiplets  $T$  which can appear has not been definitively identified. The largest number known is  $T = 192$ , for the theory with gauge group (189) [126, 131]. The structure of the anomaly lattice provides insight into the the extremal transitions where  $T$  changes; this may help in charting the connectivity and global structure of the space of F-theory models and definitively identifying the upper bound on  $T$ . Ideally such an understanding would lead to a clear picture of how the space of models arising from string/F-theory is bounded using considerations based only on the macroscopic structure of the supergravity theory.

## 5 Comments on Four Dimensions and Other Concluding Remarks

In these lectures we have described some global aspects of the space of supergravity theories in dimensions 10, 8, and 6. We have used constraints on these theories to determine a set  $\mathcal{G}$  of “apparently consistent” theories in each dimension. We have also described the set  $\mathcal{V}$  of known string theory compactifications in each dimension. We have focused on the discrete data characterizing each supergravity theory: the spectrum of fields and the gauge symmetry group of the theory. In 10D,  $\mathcal{G} = \mathcal{V}$ , so that we have “string universality”; any theory of supergravity in ten dimensions that cannot be realized in string theory is inconsistent (at least at the level of the field content and symmetry of the theory). In 8D and 6D,  $\mathcal{G} \supset \mathcal{V}$ , and the set of possible models in the “apparent swampland”  $\mathcal{G} \setminus \mathcal{V}$  is infinite. In each case, however, we have a set of fairly well-understood criteria for determining which models in  $\mathcal{G}$

admit a string realization (and hence are in  $\mathcal{V}$ ). It is possible that both in 8D and in 6D these criteria will eventually be understood in terms of additional consistency constraints on the low-energy theory; we have outlined possible approaches that may lead to such an understanding. It is also possible that new approaches to string compactification may be found that will expand the space  $\mathcal{V}$ . Whether or not it can be proven that string universality holds for supergravity theories in eight and/or six dimensions, focusing on theories in the set  $\mathcal{G} \setminus \mathcal{V}$  is a promising direction for further progress. By searching for new quantum consistency conditions that rule out such theories, or novel realizations of such theories in string theory, our understanding of the global space of quantum gravity theories is improved.

An important direction on which we have not focused in these lectures is the investigation of the relationship between macroscopic and stringy consistency conditions for more detailed aspects of supergravity theories, beyond the field content and symmetry structure. For example, for 6D  $\mathcal{N} = 1$  supergravity theories, the hypermultiplets live in a quaternionic Kähler manifold. To go beyond the discrete version of string universality, it would be necessary to demonstrate that the set of such manifolds, along with their metrics, is determined uniquely from the low-energy theory to match the set of possibilities arising from string compactifications. Compact quaternionic Kähler manifolds have been classified [159, 160, 161]. But a full analysis of which of these manifolds are compatible with string theory remains as a challenge for the future. In most supergravity theories there are also terms that might be added to the action of the theory, such as higher-derivative terms. For string realizations of these theories, the coefficients of these additional terms in the low-energy theory are fixed by the quantum physics of string theory. For string universality to hold in a complete fashion, there need to be consistency conditions that fix these coefficients to the values determined by string theory. It has been shown that in some cases, supersymmetry places very strong constraints on such coefficients [25]. But a systematic understanding of the extent to which such constraints uniquely constrain low-energy theories to those realized through string theory is another direction open for future progress.

Extending the kind of global analysis done here to four dimensions would obviously be an interesting and important endeavor. In principle this could be a very promising direction for future research. String theory constructions do impose strong constraints on which low-energy 4D supergravity theories can be realized, just as in higher dimensions. In many constructions these consistency conditions come from tadpole conditions closely analogous to the Kodaira constraint and related tadpole conditions we have described here. Anomalies have provided a clear window on these constraints in six and ten dimensions. Just as in eight dimensions, however, for four-dimensional theories some principles beyond space-time anomalies seem to be required to identify the string theoretic constraints from the structure of the low-energy theory. A number of new consistency conditions on 4D low-energy theories have been identified in recent years [1, 162, 156], though not directly related to string constructions. One natural approach that we have discussed here for deriving further constraints on theories both in 8D and in 6D is the analysis of anomalies in the world-volume of solitonic strings in supergravity theories. The same approach could be used for 4D theories. 4D supersymmetric gravity theories generally contain axions that play the

same role as  $B$  fields in higher dimensions. (Axions are dual to two-form fields in 4D.) Most if not all supersymmetric string vacuum constructions give rise to strings in 4D that couple to these axions just as strings in higher dimensions couple to two-form fields  $B_{\mu\nu}$ . Thus, a careful analysis of the world-volume theories of axionic strings in supersymmetric 4D gravity theories may give new constraints on what structures are allowed in these theories.

One complication in approaching 4D theories with less than  $\mathcal{N} = 4$  supersymmetry is that the set of string vacua is vast, and still poorly understood. Even for  $\mathcal{N} = 2$  theories there are fundamental mathematics problems that must be solved to get a global picture of the space of string vacua. Compactifications of type II string theory on Calabi-Yau threefolds (not necessarily elliptically fibered) give rise to such  $\mathcal{N} = 2$  supergravity theories. There is at this time no proof that the set of topological classes of Calabi-Yau threefolds is finite, and no systematic classification. While we were able to use the mathematical structure of lattices to great efficacy in the description of higher-dimensional supergravity theories, the analogous structure in 4D is more complex. For example, the 2-homology and intersection form on a complex surface is described by a lattice that must be self-dual due to Poincaré duality. For a general Calabi-Yau threefold, the intersection structure is given by a trilinear form  $C : (H_2(B, \mathbb{Z}))^3 \rightarrow \mathbb{Z}$ . The space of such trilinear forms, and the constraints on such forms, are much less well understood than the space of unimodular lattices. Thus, even for  $\mathcal{N} = 2$  theories in 4D, new mathematics probably must be developed to attain a true global classification of theories.

The situation becomes even more complicated for  $\mathcal{N} = 1$  theories. Studying different string approaches to  $\mathcal{N} = 1$  4D supergravity theories has been a major industry for decades. The close correspondence identified in six dimensions between the anomaly structure in supergravity theories and the topology of F-theory compactifications, along with the observation that in 6D F-theory provides a fairly universal framework for describing string vacua, suggest that F-theory is a natural place to look for a global characterization of the space of string vacua. From the F-theory point of view, however, the set of possible compactifications to 4D depends upon elliptically fibered Calabi-Yau fourfolds. Such fourfolds are fibered over a three-complex dimensional base manifold  $B$ . The classification of threefolds is in a much more limited state of development than that of surfaces. For surfaces, the minimal model approach leads to a systematic characterization of possible F-theory bases, and provides a geometric understanding of the connection between these bases. For threefolds, the program of Mori [163] aspires to a similar classification. But the mathematics is much more complicated, and seems to need significantly more development to be usable by physicists as an approach to a global characterization of the set of bases that can be used for 4D F-theory constructions. While in recent years 4D F-theory models have provided a rich new perspective on string phenomenology [164, 165, 166, 167, 168, 169, 170], the full range of possible 4D vacua is still poorly understood. Some geometric constraints on a class of 4D F-theory models have been found by Cordova [171] based on the assumption that gravity can decouple from the low-energy physics. Inclusion of fluxes (“G-flux”) in F-theory should significantly expand the range of possible 4D vacua, but the tools for understanding this from a global/nonperturbative point of view are just being developed [172, 173, 174]. As

for the 6D models discussed here,  $U(1)$  factors also have some subtle features in global constructions, and are a subject of current interest in 4D F-theory models [175], where anomaly constraints also play a role [176, 177].

Other very general approaches to building new string vacua suggest that we may have only seen the tip of the iceberg of 4D string constructions. Non-Kähler compactifications [178, 179, 180, 181, 182, 183], non-geometric fluxes [113, 184, 185, 186, 187, 188],  $G_2$  compactifications of M-theory [189, 190, 191], asymmetric orbifolds [39], and other constructions all suggest that much work must be done to begin to get some global grasp of the range of possibilities.

Finally, the discrete nature of the set of 4D  $\mathcal{N} = 1$  vacua makes it clear that attaining a global classification of models, as well as determining macroscopic constraints, will require qualitatively different insights from those used in the simpler classes of theories in higher dimensions and/or with more supersymmetry. For the 8D and 6D  $\mathcal{N} = 1$  supergravities we have studied here, the theories are connected in a single continuous moduli space of Minkowski vacua. In 4D, the inclusion of fluxes and other discrete structure can stabilize many moduli, giving a “discretium” of isolated regions of (generally AdS) supersymmetric vacua [192]. The number of discrete families of supersymmetric 4D vacua is infinite in some families of string compactifications [193]. Showing that certain apparently continuous parameters in the description of the low-energy theory can only take the specific values associated with string vacua presents an added challenge for the program of identifying constraints that rule out all models other than those realized in string theory. It may be that statistical methods [194, 195] are needed to make sense of this large set of discrete solutions. Even in 4D, however, in certain cases such as intersecting brane models on a toroidal orbifold it has been possible to place finite bounds on the set of possible vacuum solutions [196], and to systematically enumerate these models [197, 198]. Generalizing such results to more generic backgrounds such as smooth Calabi-Yau manifolds, which may be possible using the trilinear intersection form, would be a promising step forward in attaining a more global picture of large classes of vacua.

Despite the preceding cautionary remarks regarding the challenges of developing a global analysis of 4D  $\mathcal{N} = 1$  vacua, a number of the lessons learned from higher-dimensional supergravity theories should be relevant for 4D physics. Consideration of 6D theories satisfying anomaly cancellation and other constraints has suggested some new structures that may arise in string theory, such as novel types of matter from codimension two singularities in F-theory. Understanding the stringy derivation of such phenomena in six dimensions will also give new tools for describing 4D physics. In 6D, the close correspondence between the structures underlying supergravity theories and F-theory has provided a “bottom-up” map which can uniquely identify the F-theory geometry of any string vacuum realizing a supergravity model with given symmetries and spectra. Progress towards explicitly realizing such a correspondence in four dimensions would help narrow the range of string constructions which might correspond with observed physics. This systematic characterization of string vacua has also helped to identify new supergravity constraints in six dimensions, such as the self-dual nature of the dyonic charge lattice, which suggests that new constraints on 4D gravity theories remain to be discovered. Insights into the duality relationships between

different string theories in higher dimensions has played an important role in our developing understanding of string theory over the last 15 years. Incorporating these relationships into a global picture of the space of 6D string vacua will help provide insight into how the different string vacuum constructions are related in 4D.

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